Results covered in class:

Lemma 8.2 page 91. Let $\kappa$ be a regular uncountable cardinal. Then the intersection of two club subsets of $\kappa$ is also club.

Theorem 8.3 page 92. If $\kappa$ is a regular uncountable cardinal, $\lambda < \kappa$, and $\langle C_\alpha \mid \alpha < \lambda \rangle$ is a sequence of club subsets of $\kappa$, then $\bigcap_{\alpha < \lambda} C_\alpha$ is club.

Corollary to 8.3. The club filter on a regular uncountable cardinal $\kappa$ is $\kappa$-complete.

Lemma 8.4 page 92. If $\kappa$ is a regular uncountable cardinal and $\langle C_\alpha \mid \alpha < \kappa \rangle$ is a sequence of club subsets of $\kappa$, then $\bigtriangleup_{\alpha < \kappa} C_\alpha$ is also club.

Corollary 8.5 page 93. The club filter on a regular uncountable cardinal $\kappa$ is closed under diagonal intersections.

Theorem 8.7 Fodor’s Theorem page 93. Let $\kappa$ be a regular uncountable cardinal. If $f$ is a regressive function on a stationary set $S \subseteq \kappa$, then there exist stationary $T \subseteq S$ such that $f$ restricted to $T$ is constant.

Page 94: The club filter on any cardinal $\kappa \geq \omega_2$ is not an ultrafilter (consider $E^{\kappa}_{\omega}$ and $E^{\kappa}_{\omega_1}$).

Using AC and 8.10 below, the club filter on $\omega_1$ is not an ultrafilter.

Lemma 8.8. page 94. For each regular uncountable cardinal $\kappa$, every stationary subset of $E^{\omega}_\kappa$ is the union of $\kappa$ disjoint stationary sets.

Page 94. For each regular uncountable cardinal $\kappa$ and each cardinal $\lambda \in (\omega, \kappa)$, every stationary subset of $E^\lambda_\kappa$ is the union of $\kappa$ disjoint stationary sets and (using Fodor’s Theorem) every stationary subset of $\{\alpha < \kappa \mid \text{cf}(\alpha) < \alpha \}$ is the union of $\kappa$ disjoint stationary sets.

Lemma 8.9 page 94. If $S$ is a stationary subset of regular uncountable cardinal $\kappa$ such that every $\alpha \in S$ is a regular uncountable cardinal, then $T = \{\alpha \in S \mid S \cap \alpha$ is not a stationary subset of $\alpha \}$ is a stationary subset of $\kappa$.

Theorem 8.10 (Solovay) page 95. Every stationary subset of a regular uncountable cardinal $\kappa$ is the disjoint union of $\kappa$ stationary subsets.

Page 95.

1. If $\kappa$ is an inaccessible cardinal, then $\{\alpha < \kappa \mid \alpha$ is a strong limit cardinal $\}$ is club.

2. If $\kappa$ is the least inaccessible cardinal, then all strong limit cardinals below $\kappa$ are singular so that $\{\alpha < \kappa \mid \alpha$ is a singular strong limit cardinal $\}$ is club.
(3) If $\alpha < \kappa$ and $\kappa$ is the $\alpha^{th}$ inaccessible cardinal, then $\{\beta < \kappa \mid \beta$ is a regular cardinal$\}$ is nonstationary.

(4) If $\kappa$ is a Mahlo cardinal (i.e. inaccessible and $\{\alpha < \kappa \mid \alpha$ is regular$\}$ is stationary), then $\{\alpha < \kappa \mid \alpha$ is inaccessible$\}$ is stationary and $\kappa$ is the $\kappa^{th}$ inaccessible cardinal.

Theorem 9.1 Ramsey page 108. For all $n < \omega$ and all $k < \omega$, $\mathbb{S}_0 \rightarrow (\mathbb{S}_0^n)^n_k$, i.e. for every $F : [\omega]^n \rightarrow k$ there exists an infinite $H \subseteq \omega$ such that $F$ restricted to $[H]^n$ is constant.

Lemma 9.3 page 110. $2^\kappa \not\rightarrow (\omega)^2_\kappa$ for every infinite cardinal $\kappa$. In fact, the most natural partition of $[2^\kappa]^2$ into $\kappa$ pieces does not have a homogeneous set of size 3 (let alone $\omega$).

Lemma 9.4, page 110. $2^\kappa \not\rightarrow (\kappa^+)^2_2$ for every infinite cardinal $\kappa$. Therefore, $\mathbb{S}_1 \not\rightarrow (\mathbb{S}_1^2)^2_2$ and so the obvious generalization of Ramsey’s Theorem is false.

Lemma 9.5. page 110. The lexicographically ordered set $\{0,1\}^\kappa$ has no increasing or decreasing $\kappa^+$-sequence. (This is used to prove Lemma 9.4.)

Discussed Ramsey cardinals from page 121.

Lemma 8.11 page 96. If $\kappa$ is regular and uncountable and if $F$ is a normal filter that contains (as elements) all final segments $(\alpha,\kappa)$, then $F$ contains all club subset of $\kappa$.

Lemma 10.4 page 127. Every measurable cardinal is inaccessible (i.e. regular and a strong limit cardinal).

Lemma 10.19. page 134 If $D$ is a normal measure on $\kappa$ (i.e. a $\kappa$-complete nonprincipal ultrafilter closed under diagonal intersections), then every set in $D$ is stationary.

Theorem 10.20 page 134. Every measurable cardinal carries a normal measure. If $U$ is a nonprincipal $\kappa$-complete ultrafilter on $\kappa$, then there exists a function $f : \kappa \rightarrow \kappa$ such that $f : (U) = \{X \subseteq \kappa \mid f^{-1}(X) \in U\}$ is a normal measure.

Theorem 10.21 page 135. Every measurable cardinal is a Mahlo cardinal (i.e. inaccessible and $\{\alpha < \kappa \mid \alpha$ is regular$\}$ is stationary). In fact, $\{\alpha < \kappa \mid \alpha$ is inaccessible$\}$ is an element of the normal measure and therefore stationary by 10.19.

Theorem 10.22. page 136. Every measurable cardinal is a Ramsey cardinal. In fact, if $D$ is a normal measure on a measurable cardinal $\kappa$, then for every $F : [\kappa]^\omega \rightarrow \lambda$ with $\lambda < \kappa$ there exists $H \in D$ that is homogeneous for $F$ (i.e. $\forall n < \omega \exists i_n < \lambda$ $F$ only takes the constant value $i_n$ on $[H]^n$).