A. Motivation

1. Bayes’ Theorem is a special application of conditional probability. An example leads to the theorem:

2. Surf Mart, which sells shirts under its’ own label, buys 40% of its shirts from supplier A, 50% from supplier B, and 10% from supplier C. It is found that 2% of the shirts from A have flaws, 3% from B have flaws and 5% from C have flaws. A probability tree diagram is shown below for you to complete. If one of the shirts is bought from Surf Mart, what is the probability that:

   (a) the shirt has a flaw, given that it came from A? Knowing that the shirt came from A means that we have already traversed the A branch and now look for the probability, which is 0.02, along the attached flow branch. Therefore \( P(\text{flaw}|\text{from A}) \) is 0.02.

   (b) The shirt has a flaw? \( P(\text{from A and a flaw or from B and a flaw or from C and a flaw}) = (0.4)(0.02) + (0.5)(0.03) + (0.1)(0.05) = 0.028 \)

   (c) The shirt came from A given that it has a flaw? This time, the given event is on the right and we are confronted with somehow trying to read back to the left to reach the event A. This is the reverse probability to which referred and will always be a clue that the problem is a Bayes Theorem problem. We find the solution:

\[
P(A|\text{flaw}) = \frac{P(A \text{ and flaw})}{P(\text{flaw})} = \frac{(0.40)(0.02)}{(0.4)(0.02) + (0.5)(0.03) + (0.1)(0.05)} = \frac{0.008}{0.028} = 0.286
\]

B. Bayes Theorem

1. Partition: A sample space \( S \) is partitioned into \( n \) subsets \( A_1, A_2, \ldots, A_n \), provided

   (a) Each subset is nonempty

   (b) The intersection of any two of the subsets is empty
(c) \( A_1 \cup A_2 \cup \cdots \cup A_n = S \)

2. Let \( S \) be a sample space and let \( A_1, A_2, \ldots, A_n \) be \( n \) events that form a partition of the set \( S \). If \( E \) is any event in \( S \), then

\[
E = (E \cap A_1) \cup (E \cap A_2) \cup \cdots \cup (E \cap A_n)
\]

Since \( E \cap A_1, E \cap A_2, \ldots, E \cap A_n \) are mutually exclusive events, we have

\[
P(E) = P(E \cap A_1) + P(E \cap A_2) + \cdots + P(E \cap A_n)
\]

Now, if we replace \( P(E \cap A_1), P(E \cap A_2), \ldots, P(E \cap A_n) \) using the Product Rule. Then we obtain the formula

\[
P(E) = P(A_1) \cdot P(E | A_1) + P(A_2) \cdot P(E | A_2) + \cdots + P(A_n) \cdot P(E | A_n)
\]

3. Bayes Theorem: Let \( S \) be a sample space partitioned into \( n \) events, \( A_1, \ldots, A_n \). Let \( E \) be any event of \( S \) for which \( P(E) > 0 \). The probability of the event \( A_j \) \((j = 1, 2, \ldots, n)\), given the event \( E \), is

\[
P(A_j | E) = \frac{P(A_j) \cdot P(E | A_j)}{P(E)} = \frac{P(A_j) \cdot P(E | A_j)}{P(A_1) \cdot P(E | A_1) + P(A_2) \cdot P(E | A_2) + \cdots + P(A_n) \cdot P(E | A_n)}
\]