PC-Labeling of a Graph and its PC-Set

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Abstract

A binary vertex labeling \( f : V(G) \to \mathbb{Z}_2 \) of a graph \( G \) is said to be friendly if the number of vertices labeled 0 is almost the same as the number of vertices labeled 1. This friendly labeling induces an edge labeling \( f^* : E(G) \to \mathbb{Z}_2 \) defined by \( f^*(uv) = f(u)f(v) \) for all \( uv \in E(G) \). Let \( e_f(i) = \{uv \in E(G) : f^*(uv) = i\} \) be the number of edges of \( G \) that are labeled \( i \). Product-cordial index of the labeling \( f \) is the number \( pc(f) = |e_f(0) - e_f(1)| \). The product-cordial set of the graph \( G \), denoted by \( PC(G) \), is defined by

\[
PC(G) = \{pc(f) : f \text{ is a friendly labeling of } G \}.
\]

In this paper we will determine the product-cordial sets of certain classes of graphs.

Key Words: friendly labeling, product-cordial index, product-cordial set.

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1 Introduction

In this paper all graphs \( G = (V, E) \) are connected, finite, simple, and undirected. For graph theory notations and terminology not described in this paper, we refer the readers to [6]. Let \( G \) be a graph and \( f : V(G) \to \mathbb{Z}_2 \) be a binary vertex coloring (labeling) of \( G \). For \( i \in \mathbb{Z}_2 \), let \( v_f(i) = |f^{-1}(i)| \). The coloring \( f \) is said to be friendly if \( |v_f(1) - v_f(0)| \leq 1 \). That is, the number of vertices colored 0 is almost the same as the number of vertices colored 1.

Any friendly coloring \( f : V(G) \to \mathbb{Z}_2 \) induces an edge labeling \( f^* : E(G) \to \mathbb{Z}_2 \) defined by \( f^*(xy) = f(x)f(y) \) \( \forall xy \in E(G) \). For \( i \in \mathbb{Z}_2 \), let \( e_f(i) = |f^{*-1}(i)| \) be the number of edges of \( G \) that are labeled \( i \). The number \( pc(f) = |e_f(1) - e_f(0)| \) is called the product-cordial index (or pc-index) of \( f \). The product-cordial set (or pc-set) of the graph \( G \), denoted by \( PC(G) \), is defined by

\[
PC(G) = \{pc(f) : f \text{ is a friendly vertex coloring of } G \}.
\]

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To illustrate the above concepts, consider the graph $G$ of Figure 1, which has six vertices. The condition $|v_f(1) - v_f(0)| \leq 1$ implies that three vertices be labeled 0 and the other three 1.

$$G: \begin{array}{ccc}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 0 & 0 \\
\end{array}$$

Figure 1: An example of product-cordial labeling of $G$.

Figure 1 also shows the associated edge labeling of $G$, where one edge has label 1 while the other 9 edges have labels 0. Therefore, the product-cordial index (or pc-index) of this labeling is $9 - 1 = 8$. It is easy to see that $PC(G) = \{4, 6, 8\}$. The friendly colorings of $G$ that provide the other two pc-indices are presented in Figure 2.

$$\begin{array}{ccc}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 0 & 0 \\
\end{array} \quad \begin{array}{ccc}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{array}$$

Figure 2: Two friendly labelings of $G$ with pc-indices 4 and 6.

For a graph $G = (p, q)$ of size $q$, and a friendly labeling $f : V(G) \to \mathbb{Z}_2$ of $G$, we have

$$pc(f) = |e_f(0) - e_f(1)| = |q - 2e_f(1)|. \quad (1.1)$$

Therefore, to find the pc-index of $f$ it is enough to find $e_f(1)$. Moreover, to determine the pc-set of $G$ it is enough to compute $e_f(1)$ for different friendly labelings. Another immediate consequence of (1.1) is the following useful fact:

**Observation 1.1.** For a graph $G$ of size $q$, $PC(G) \subseteq \{q - 2k : 0 \leq k \leq \lfloor q/2 \rfloor\}$.

**Definition 1.2.** A graph $G$ of size $q$ is said to be fully product-cordial (or fully pc) if

$$PC(G) = \{q - 2k : 0 \leq k \leq \lfloor q/2 \rfloor\}.$$ 

For example, the graph $G$ of Figure 1 is not fully pc. However, $P_n$, the path of order $n$, is fully pc. In case of $P_n$, it is easy to observe that $e_f(1) = 0, 1, \cdots, \lfloor \frac{n-1}{2} \rfloor$, which proves that
Theorem 1.3. For any $n \geq 2$, $PC(P_n) = \{n - 1 - 2k : 0 \leq k \leq \lfloor \frac{n-1}{2} \rfloor \}$.

The different friendly labelings of $P_7$ that provide its pc-set are illustrated in Figure 3.

Figure 3: $PC(P_7) = \{0, 2, 4, 6\}$.

In 1978, I. Cahit [2, 3, 4] introduced the concept of cordial labeling as a weakened version of the less tractable graceful and harmonious labeling. Given a cordial labeling $f : V(G) \rightarrow \mathbb{Z}_2$ of a graph $G$, Cahit introduced an edge labeling $f^+ : E(G) \rightarrow \mathbb{Z}_2$ by $f^+(uv) = |f(u) - f(v)|$ and defined the cordial index $c(f)$ of $f$ to be $|f^{-1}(0) - f^{+1}(1)|$. A graph is called cordial if it admits a cordial labeling with cordial index 0 or 1. Cahit, among other facts, proved that

1. every tree is cordial;
2. The complete graph $K_n$ is cordial if and only if $n \leq 3$;
3. The complete bipartite graph $K(m,n)$ is cordial ($m, n \in \mathbb{N}$);
4. The wheel $W_n$ is cordial if and only if $n \equiv \not\equiv 3 \ (mod \ 4)$;
5. In an Eulerian graph $G = (p, q)$ if $p \equiv 0 \ (mod \ 4)$, then it is not cordial.

M. Hovay [9], later generalized the concept of cordial graphs and introduced $A$-cordial labelings. A graph $G$ is said to be $A$-cordial if it admits a labeling $f : V(G) \rightarrow A$ such that for every $i, j \in A$,

$$|v_f(i) - v_f(j)| \leq 1 \text{ and } |e_f(i) - e_f(j)| \leq 1.$$ 

Cordial graphs have been studied extensively. Interested readers are referred to a number of relevant literature that are mentioned in the bibliography section, including [1, 5, 8, 10, 11, 14, 18].

Product cordial labeling of a graph was introduced by Sundaram, Ponraj and Somasundaran [21]. They call a graph $G$ product-cordial if it admits a friendly labeling whose product-cordial index is at most 1. Then Sundaram, Ponraj and Somasundaran [21, 22, 23] investigate whether certain graphs such as trees, cycles, complete graphs, wheels, etc. are product-cordial. In this paper the product-cordial sets of certain classes of graphs are determined. Naturally, if $0 \in PC(G)$ or $1 \in PC(G)$, then $G$ would be product-cordial.
2 PC Sets of Complete Graphs

Theorem 2.1. For any \( n \geq 3 \), \( PC(K_n) = \{|n/2|^2, |n/2|^2 + |n/2|(1 - (-1)^n)\} \).

Proof. Let \( f : V(K_n) \rightarrow \mathbb{Z}_2 \) be an arbitrary friendly labeling of \( K_n \). We consider the following two cases:

(a) Let \( n = 2\ell \) be even. Then \( \ell \) vertices of \( K_n \) are labeled 1. Therefore, the induced edge labeling marks exactly \( \ell(\ell - 1)/2 \) edges of \( K_n \) by 1. That is, \( e_f(1) = \ell(\ell - 1)/2 \). Hence \( pc(f) = \ell(2\ell - 1) - \ell(\ell - 1) = \ell^2 = |n/2|^2 \).

(b) Let \( n = 2\ell + 1 \) be odd. Then either \( \ell + 1 \) or \( \ell \) vertices of \( K_n \) are labeled 1. Therefore, \( e_f(1) = \ell(\ell+1)/2 \) or \( \ell(\ell-1)/2 \). Hence \( pc(f) = \ell^2 = |n/2|^2 \) or \( \ell^2+2\ell = |n/2|^2 + |n/2|(1 - (-1)^n) \).

\( \Box \)

Theorem 2.2. For any \( n \geq 4 \), \( PC(K_n - e) = \{|n/2|^2 \pm 1, |n/2|^2 \pm 1 + (1 - (-1)^n)|n/2|\} \).

Proof. It is easy to verify that \( PC(K_4 - e) = \{3, 5\} \). The two friendly coloring that provide these two numbers are given in Figure 4.

\[ \begin{array}{c}
\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{array}
\end{array} \]

Figure 4: \( PC(K_4 - e) = \{3, 5\} \).

Assume \( n \geq 5 \). Let \( u, v \) be the two vertices of the edge \( e \), and let \( f : V(K_n - e) \rightarrow \mathbb{Z}_2 \) be an arbitrary friendly labeling of \( K_n - e \). We consider the following two cases:

Case I. \( n = 2\ell \). Then \( \ell \) vertices of \( K_n \) are labeled 1. If \( f(u) = 0 \) or \( f(v) = 0 \), then \( K_n \) has \( K_\ell \) as a subgraph all of whose vertices are labeled 1. That is, the induced edge labeling of \( f \) marks exactly \( |K_\ell| = \ell(\ell - 1)/2 \) edges of \( K_n \) by 1. Therefore, \( e_f(1) = \ell(\ell - 1)/2 \).

If \( f(u) = f(v) = 1 \), then \( K_n \) has \( K_\ell - e \) as a subgraph all of whose vertices are labeled 1. That is, the induced edge labeling of \( f \) marks exactly \( |K_\ell - e| \) edges of \( K_n \) by 1. Therefore, \( e_f(1) = -1 + \ell(\ell - 1)/2 \).

In this case, the only two elements of \( PC(K_n) \) would be \( \ell^2 \pm 1 \).

Case II. Let \( n = 2\ell + 1 \) be odd. In this case we have two options:

(a) \( \ell \) vertices of \( K_n \) are labeled 1 and the other \( \ell + 1 \) vertices are labeled 0. Then, by the argument presented in case I, \( e_f(1) = \ell(\ell - 1)/2 \) or \( e_f(1) = -1 + \ell(\ell - 1)/2 \).

(b) \( \ell + 1 \) vertices of \( K_n \) are labeled 1 and the other \( \ell \) vertices are labeled 0. Then, by a similar argument that is presented in case I, \( e_f(1) = \ell(\ell + 1)/2 \) or \( e_f(1) = -1 + \ell(\ell + 1)/2 \).

Therefore, in this case, there are exactly four elements in \( PC(K_n) \), which are \( \ell^2 \pm 1 \) and \( \ell^2+2\ell \pm 1 \). \( \Box \)
Theorem 2.3. The pc-set of any bipartite graph $K_{m,n}$ is
\[ \left\{ mn + 2i^2 - i(m + n) \pm \frac{i}{2} \left( 1 - (-1)^{m+n} \right) : 0 \leq i \leq \left\lfloor \frac{m + n + 1}{4} \right\rfloor \right\}. \]

Proof. Let $S = \{u_1, u_2, \ldots, u_m\}$ and $T = \{v_1, v_2, \ldots, v_n\}$ be the two partite sets. Due to symmetry of $K_{m,n}$, we may assume $m \leq n$. The graph $K_{m,n}$ has $m + n$ vertices and $mn$ edges. Consider the friendly labeling $f : V(K_{m,n}) \rightarrow \mathbb{Z}_2$ of $K_{m,n}$ defined by $f(u_1) = f(u_2) = \cdots = f(u_i) = 1$, $f(v_1) = f(v_2) = \cdots = f(v_j) = 1$, and label the remaining vertices by 0. We consider the following two cases:

(a) Let $m + n$ be even. Then $2(i + j) = m + n$ and $e_f(1) = ij$. Therefore, $pc(f) = mn - 2ij = mn + 2i^2 - i(m + n)$, where $i = 0, 1, 2, \ldots, m$.

(b) Let $m+n$ be odd. In this case $2(i+j) = m+n\pm1$ and $pc(f) = mn-2ij = mn+2i^2-i(m+n)\pm i$, where $i = 0, 1, 2, \ldots, m$.

Finally, we observe that if we define $h(i) = mn + 2i^2 - i(m + n)$, then $h\left(\frac{m+n}{2} - i\right) = h(i)$. Therefore, to generate all the elements of $PC(K_{m,n})$, it is enough to let $i = 0, 1, \cdots, \left\lfloor \frac{m+n+1}{4} \right\rfloor$. \hfill \qed

3 PC-Sets of Trees

The complete bipartite graphs $K_{1,n}$ is also known as stars. By theorem 2.3 we know that:

Theorem 3.1. $PC(K_{1,n}) = \begin{cases} \{1, n\} & \text{if } n \text{ is odd;} \\ \{0, 2, n\} & \text{if } n \text{ is even.} \end{cases}$

Three different friendly labelings of $K_{1,6}$ that provide its pc-set are illustrated in Figure 5.

![Figure 5: PC($K_{1,6}$) = \{0, 2, 6\}.](image)

A double star is a tree of diameter 3. Double stars have two central vertices $u$ and $v$ and are denoted by $DS(m, n)$, where $\deg u = m$ and $\deg v = n$, as illustrated in Figure 6.

Theorem 3.2. The pc-set of double-star $DS(m, n)$ with $m \leq n$ is
\[ \left\{ m + n - 1 - 2k : 0 \leq k \leq m - 1 \text{ or } \left\lfloor \frac{n - m}{2} \right\rfloor \leq k \leq \left\lfloor \frac{m + n - 1}{2} \right\rfloor \right\}. \]

Proof. Consider $G = DS(m, n)$, which has $m + n$ vertices, hence $m + n - 1$ edges. Note that $DS(1, 1) \cong P_2$, $DS(1, 2) \cong DS(2, 1) \cong P_3$, $DS(1, n) \cong DS(n, 1) \cong ST(n)$ and their pc-numbers are
calculated in theorems 1.3 and 3.1. Furthermore, the results are consistent with the statement of this theorem.

Assume that $2 \leq m \leq n$ and let $f : V(G) \to \mathbb{Z}_2$ be a friendly labeling of $G$ with $f(u_1) = f(u_2) = \cdots = f(u_i) = 1$ and $f(v_1) = f(v_2) = \cdots = f(v_j) = 1$. For the labels of the central vertices $u$ and $v$ we have the following four options:

**Option 1.** Let $f(u) = f(v) = 0$. Then $e_f(1) = 0$ and $pc(f) = m + n - 1$.

**Option 2.** Let $f(u) = f(v) = 1$. Then $e_f(1) = i + j + 1$ and $pc(f) = m + n - 1 - 2(i + j + 1)$. Now if $m + n$ is even, then $2(i + j + 2) = m + n$ and therefore $pc(f) = 1$. However, if $m + n$ be odd, then $2(i + j + 2) = m + n \pm 1$, which results in $pc(f) = 0, 2$.

**Option 3.** Let $f(u) = 1$, $f(v) = 0$. Then $e_f(1) = i$ and $pc(f) = m + n - 1 - 2i$, with $0 \leq i \leq m - 1$.

**Option 4.** Let $f(u) = 0$, $f(v) = 1$. Then $e_f(1) = j$ and $pc(f) = m + n - 1 - 2j$, with $\lfloor (m - n)/2 \rfloor \leq j \leq \lfloor (m + n - 1)/2 \rfloor$.

These four options imply that the pc-set of $DS(m, n)$ is a subset of $\{m + n - 1 - 2k : 0 \leq k \leq m - 1\}$ or $\lfloor (n - m)/2 \rfloor \leq k \leq \lfloor (m + n - 1)/2 \rfloor$. On the other hand, every friendly labeling of $DS(m, n)$ is among one of the above options. Therefore, the equality holds.

**Corollary 3.3.** The double-star $DS(m, n)$ is fully pc if and only if $n \leq 3m + 1$.

**Proof.** We note that $DS(m, n)$ is fully pc if and only if $m \geq \lfloor (n - m)/2 \rfloor$, which implies that $n \leq 3m + 1$. Conversely, if $n \leq 3m + 1$, then $(n - m)/2 \leq m + 1/2$ and $\lfloor (n - m)/2 \rfloor \leq m$. □

**Examples 3.4.**

(a) The graph $DS(m, m)$ has $2m$ vertices and $2m - 1$ edges. Therefore, $PC(DS(m, m)) = \{2m - 1 - 2k : 0 \leq k \leq m - 1\} = \{1, 3, \ldots, 2m - 1\}$.

(b) The graph $DS(m, m + 1)$ has $2m$ edges. Hence, $PC(DS(m, m + 1)) = \{2m - 2k : 0 \leq k \leq m\} = \{0, 2, \ldots, 2m\}$.

(c) $PC(DS(3, 7)) = \{9 - 2k : 0 \leq k \leq 4\} = \{1, 3, 5, 7, 9\}$, and $DS(3, 7)$ is fully product cordial.
(d) $PC(DS(3, 13)) = \{15 - 2k : 0 \leq k \leq 2 \text{ or } 5 \leq k \leq 7\} = \{1, 3, 5, 11, 13, 15\}$ and $DS(3, 13)$ is not fully product cordial.

4 PC Sets of Cycles and Cycle Related Graphs

Theorem 4.1. For any $n \geq 3$, the graph $C_n$ is fully product-cordial. That is, $PC(C_n) = \{n - 2k : 0 \leq k \leq \lfloor \frac{n-1}{2} \rfloor \}$.

Proof. Let $v_1, v_2, \cdots, v_n$ be the vertices of $C_n$. We consider two cases:

(a) $n = 2\ell$. We show that for any $j \in \{0, 1, \cdots, \ell - 1\}$ there is a friendly labeling of $C_n$ whose pc-index is $j$. To achieve this, we label $\ell + 1$ consecutive vertices $v_1, v_2, \cdots, v_{j+1}$ by 1 then alternate the labels 0 and 1 and label all the remaining vertices by 0. The induced edge labeling labels $j$ edges with 1 and the rest with 0.

(b) $n = 2\ell + 1$. With a similar argument, one can show that for any $j \in \{0, 1, \cdots, \ell\}$ there is a friendly labeling of $C_n$ whose pc-index is $j$.

Three different friendly labelings of $C_6$ that provide its pc-set are illustrated in Figure 7.

![Figure 7: PC($C_6$) = \{2, 4, 6\}](image)

For $n \geq 3$, the wheel $W_n$ is the graph $C_n + K_1$.

Theorem 4.2. For each integer $n \geq 3$,

$PC(W_n) = \left\{2n - 2j : 1 - (-1)^n \leq j \leq 2\lfloor n/2 \rfloor - 1\right\}$.

Proof. Construct $W_n = C_n + K_1$ from an $n$-cycle $C_n : u_1, u_2, \cdots, u_n, u_1$ by adding a new vertex $v$ (central vertex) and joining $v$ to every vertex of $C_n$. Note that $W_n$ has $n + 1$ vertices and $2n$ edges. First, we observe that the maximum value of $e_f(1)$ occurs when the central vertex is labeled 1 and $\lfloor n/2 \rfloor$ consecutive vertices of $C_n$ are labeled 1. The induced edge labeling of this particular friendly labeling marks $2\lfloor n/2 \rfloor - 1$ edges of $W_n$ by 1. Therefore, for any friendly labeling $f : V(W_n) \to \mathbb{Z}_2$, we have $e_f(1) \leq 2\lfloor n/2 \rfloor - 1$. Now, based on the parity of $n$, we consider the following two cases:

Case 1. The cycle has even number of vertices. In this case, the statement of the Theorem becomes $PC(W_{2n}) = \{4n - 2j : 0 \leq j \leq 2n - 1\}$. We now show that for every $j = 0, 1, 2, \cdots, 2n - 1$, there is a friendly labeling $f : V(W_{2n}) \to \mathbb{Z}_2$ with $e_f(1) = j$. 

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Subcase 1A. $0 \leq j \leq n - 1$. We define $f : V(W_{2n}) \to \mathbb{Z}_2$ by $f(v) = 0$ and

$$f(u_i) = \begin{cases} 1 & \text{if } 1 \leq i \leq j + 1; \\ (1 - (-1)^i + j)/2 & \text{if } j + 1 < i < 2n - j; \\ 0 & \text{if } 2n - j \leq i \leq 2n. \end{cases}$$

This coloring labels $n$ vertices of $C_{2n}$ by 1 and the other $n$ vertices of $C_{2n}$ by 0. Hence, it is a friendly labeling of $W_{2n}$. The induced edge labeling of this function marks exactly $j$ edges of $W_{2n}$ by 1. That is, $e_f(1) = j$.

Subcase 1B. $n \leq j \leq 2n - 1$. Let $g : V(W_{2n}) \to \mathbb{Z}_2$ be the inverse coloring of $f$ defined by $g(x) = 1 - f(x)$ $\forall x \in V(W_{2n})$. The coloring $g$ labels $n + 1$ vertices of $W_{2n}$ by 1 and the other $n$ vertices by 0. Hence, $g$ is a friendly labeling. Moreover, $pc(g) = n + pc(f)$. When $pc(f)$ changes from 0 to $n - 1$, then $pc(g)$ covers the numbers $n$ through $2n - 1$. This completes the proof in this case.

Six different friendly labelings of $W_6$ that provide its pc-set are illustrated in Figure 8, the labelings in each row are the inverse labelings of the other row.

![Figure 8](image_url)

Case 2. The cycle has odd number of vertices. First, we observe that the minimum value of $PC(W_{2n+1})$ occurs when the central vertex $v$ and $n$ vertices of the cycle $C_{2n+1}$ are labeled 0. As a result, the other $n + 1$ vertices of $C_{2n+1}$ get label 1. Consequently, at least two adjacent vertices of $C_{2n+1}$ get label 1 and the induced edge labeling of this cordial labeling marks at least one edge of $W_{2n+1}$ with 1. Therefore, for any friendly labeling $f : V(W_{2n+1}) \to \mathbb{Z}_2$ we have $1 \leq e_f(1) \leq 2n - 1$.

Next, note that in this case, the statement of the Theorem becomes $PC(W_{2n+1}) = \{4n + 2 - 2j : 1 \leq j \leq 2n - 1\}$. To prove this, a similar scheme that was used in case 1 can be utilized to show that for every $j = 1, 2, \cdots, 2n - 1$, there is a friendly labeling $f : V(W_{2n+1}) \to \mathbb{Z}_2$ with $e_f(1) = j$.

Three different friendly labelings of $W_5$ that provide its pc-set are illustrated in Figure 9.

Corollary 4.3. For $n \geq 3$, the wheel $W_n$ is not fully product-cordial.
Figure 9: $PC(W_5) = \{4, 6, 8\}$.

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References


