ON $P_3$-DEGREE OF GRAPHS

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Abstract. It is known that there is not any non-trivial graph with vertices of distinct degrees, and any non-trivial graph must have at least two vertices of the same degree. In this article, we will consider the concept of $P_3$-degree of vertices and will introduce a class of connected graphs with exactly two vertices of the same $P_3$-degree. Also, the graphs with distinct $P_3$-degree vertices will be constructed and it will be proven that for any $n \geq 6$ there is at least one graph of order $n$, with distinct $P_3$-degree vertices.

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1. Introduction

Unless otherwise stated all graphs considered are connected, finite, simple, undirected, and of order $n \geq 3$. It is known that there is not a graph with vertices of distinct degrees, and any non-trivial graph must have at least two vertices of the same degree. Behzad-Chartrand [2] and Nebesky [7] have studied the graphs containing exactly two vertices of the same degree. Also, Kac-Nesterova have investigated the graphs with exactly three vertices of the same degree. In this article, just for the sake of brevity, a graph $G$ is said to be pairlone if there are precisely two vertices with the same degree. In Figure 1, three examples of pairlone graphs of different orders are illustrated, and each vertex is labeled by its degree. Even though the existence of pairlone graphs has been shown in [2, 7], for the sake of completeness and for later use, we give the proof of the following theorem.

Theorem 1.1. For any $n \geq 2$ there is a unique pairlone graph of order $n$. Furthermore, the degree sequence is 1 through $n - 1$ with $\lfloor n/2 \rfloor$ appearing twice.
Figure 1. Three examples of pairlone graphs.

Proof. We proceed by induction on \( n \), case of \( n = 2 \) is obvious. Now let \( G \) be the unique connected pairlone graph of order \( n \geq 2 \), that has degree sequence 1 through \( n - 1 \) with \( \lfloor n/2 \rfloor \) appearing twice. That is, \( V(G) = \{u_1, u_2, \cdots, u_n\} \) and
\[
\deg_G u_i = \begin{cases} 
  i & \text{for } i \leq \lfloor n/2 \rfloor \\
  i - 1 & \text{for } i > \lfloor n/2 \rfloor
\end{cases}
\]
We use \( G \) to construct a graph \( H \) by adding a vertex \( u \), \( V(H) = V(G) \cup \{u\} \), and edges \( uu_k \ [n/2] < k \leq n \), as illustrated in Figure 2. With this construction, \( \deg_H u_i = i \ (1 \leq i \leq n) \), and \( \deg_H u = n - \lfloor n/2 \rfloor = \lceil n+1/2 \rceil \).

Therefore, \( H \) is a connected graph of order \( n + 1 \), which has degree sequence 1 through \( n \) with \( \lfloor n+1/2 \rfloor \) repeated twice. The uniqueness of such graph follows from the fact that in any connected pairlone graph of order \( n + 1 \), removal of one of the vertices of order \( \lfloor n+1/2 \rfloor \) will result in the unique connected pairlone graph of order \( n \). \( \square \)

In what follows \( PL_n \) stands for the unique connected pairlone graph of order \( n \). Let \( v_1, v_2, \cdots, v_n \) be the vertices of \( PL_n \) labeled so that \( \deg v_1 \leq \deg v_2 \leq \cdots \leq \deg v_n \).

Observation 1.2. The chromatic number of \( PL_n \), the unique connected pairlone graph of order \( n \), is \( n \) – \( \lfloor n-1/2 \rfloor \).
**Proof.** We observe that
\[
\text{deg } v_i = \begin{cases} 
  i & \text{for } i \leq \lfloor n/2 \rfloor; \\
  i - 1 & \text{for } i > \lfloor n/2 \rfloor,
\end{cases}
\]
and the edges of \(G\) consist of all \(v_{n-i}v_j\) with \(i = 0, 1, \ldots, \lfloor n/2 \rfloor - 1\), and \(i + 1 \leq j \leq n - i - 1\). Admissibility of the vertices of the same degree \(\lfloor n/2 \rfloor\) depends on the parity of \(n\). When \(n\) is even, they are adjacent and if \(n\) is odd they are not. We also observe that the vertices \(v_1, v_2, \ldots, v_{\lfloor n+1/2 \rfloor}\) are not adjacent, while the vertices \(v_n, v_{n-1}, \ldots, v_{\lfloor n+1/2 \rfloor}\) are pairwise adjacent. Consequently, the subgraph of \(PL_n\) induced by \(p = n - \lfloor n-1/2 \rfloor\) vertices \(v_n, v_{(n-1)}, \ldots, v_{\lfloor n+1/2 \rfloor}\) is isomorphic to the complete graph \(K_p\). Therefore, \(\chi(PL_n) \geq p\). On the other hand, in any coloring \(c : V(PL_n) \rightarrow \mathbb{N}\) of \(PL_n\), since \(v_1, v_2, \ldots, v_{\lfloor n+1/2 \rfloor}\) are not adjacent, they can be assigned the same color; Specifically, \(c(v_i) = \begin{cases} 
  n - \lfloor n-1/2 \rfloor & \text{for } 1 \leq i \leq \lfloor n+1/2 \rfloor; \\
  n+1 - i & \text{for } \lfloor n+1/2 \rfloor \leq i \leq n,
\end{cases}\) which implies that \(\chi(PL_n) \leq p\). Therefore, \(\chi(PL_n) = p\). For the pairlone graphs \(PL_5\) and \(PL_7\), this coloring is illustrated in Figure 3.

![Figure 3](image)

**Figure 3.** \(\chi(PL_5) = 3\) and \(\chi(PL_7) = 4\).

### 2. \(P_3\)-Pairlone Graphs

The notion of degree of a vertex in a graph has been generalized in [4] by introducing \(F\)-degree in the following way: For a given graph \(F\), the \(F\)-degree of a vertex \(v\) in \(G\), denoted by \(F\)-deg\((v)\), is the number of subgraphs of \(G\) isomorphic to \(F\) that contain \(v\). A concept related to the \(F\)-degree of a vertex was first introduced by Kocay [6], when reconstructing degree sequence on graphs. A graph \(G\) is said to be \(F\)-regular if the \(F\)-degrees of all the vertices of \(G\) are the same and it is called \(F\)-irregular if the \(F\)-degrees of the vertices of \(G\) are distinct. We observe that the ordinary degree of a vertex \(v\) is the \(P_2\)-degree \((F = P_2)\) of \(v\), and when we consider \(P_2\)-degrees there is not any irregular graph. In this section, we will focus on the \(P_3\)-degree of a graph and will show that, as opposed to ordinary degree, for \(n \geq 6\) there is at least one graph of order \(n\) which is
$P_3$-irregular. Also, the existence of $P_3$-pairlone graphs that are not (ordinary degree) pairlone will be proven.

**Definition 2.1.** Given a graph $G$, the $P_3$-degree of a vertex $v$ in $G$, denoted by $P_3$-deg$(v)$, is the number of subgraphs of $G$ isomorphic to $P_3$ that contain $v$. The graph $G$ is said to be $P_3$-regular if the $P_3$-degrees of all the vertices are the same, and $G$ is said to be $P_3$-irregular if the $P_3$-degrees of its vertices are distinct.

To illustrate the definition of $P_3$-degree, consider the graph in Figure 4.

![Figure 4](image.png)

**Figure 4.** $P_3 - \deg(v) = 5$.

Vertices of $G$ are labeled by their $P_3$-degrees; For example, there are 5 subgraphs of $G$ isomorphic to $P_3$ that contain the vertex $v$. Therefore, $P_3 - \deg(v) = 5$. As we will see later 2.5, there are infinitely many $P_3$-degree irregular graphs. Figure 5 illustrate one of them for which the $P_3$-degrees of vertices are identified.

![Figure 5](image.png)

**Figure 5.** $P_3$-degree irregular graph of order 6.

We note that a vertex can occur either as the central vertex or as an end-vertex of $P_3$. Now let $v \in V(G)$ and let $N(v)$ be the set of all vertices of $G$ adjacent to $v$. Then $v$ is the central vertex of $\binom{\deg v}{2}$ subgraphs of $G$ isomorphic to $P_3$ and is an end-vertex of $\sum_{u \in N(v)} (\deg u - 1)$ such subgraphs.

Therefore

$$P_3 - \deg v = \binom{\deg v}{2} + \sum_{u \in N(v)} (\deg u - 1).$$

(2)

For regular graphs we have the following theorem [4]:

48
Theorem 2.2. A graph $G$ is $P_3$-regular of degree $k \geq 2$ if and only if $G$ is regular of degree $r \geq 2$, where $k = 3{r \choose 2}$.

![Figure 6. Three $P_3$-degree pairlone graphs.](image)

Theorem 2.3. For any $n \geq 4$, the graph $PL_n$ is $P_3$-degree pairlone.

Proof. Let $PL_n$ be the unique connected pairlone graph of order $n$. Using (1) and (2), we notice that

$$P_3 - \text{deg}_i = \begin{cases} i(n-2) & \text{for } i \leq \lfloor n/2 \rfloor; \\ (i-1)(n-2) + \lfloor n/2 \rfloor + 1 - i & \text{for } i > \lfloor n/2 \rfloor. \end{cases}$$

Thus vertices of $PL_n$ have distinct $P_3$-degrees, except for exactly one pair having each vertex the same ordinary degree $\lfloor n/2 \rfloor$. \hfill \Box

Theorem 2.4. For any $n \geq 5$ there is at least one $P_3$-pairlone graph of order $n$, that is not (ordinary degree) pairlone.

Proof. Let $G = PL_n$ be the unique connected pairlone graph of order $n$. We construct $H$ from $PL_n$ by adding a vertex $v_0$ to $V(G)$ and the edge $v_0v_1$ to $E(G)$, as illustrated in Figure 7, for $n = 4, 5, 6$. Then, as a result of this modification, $H$ will have another subgraph with vertices $v_0, v_1,$ and $v_n$, isomorphic to $P_3$ (with $v_1$ as its central vertex, and $v_0, v_n$ as its end-vertices). Thus $P_3 - \text{deg} v_0 = 1$, and

$$P_3 - \text{deg}_H v_i = \begin{cases} P_3 - \text{deg}_G v_i & \text{for } i \neq 1, n; \\ 1 + P_3 - \text{deg}_G v_i & \text{for } i = 1, n. \end{cases}$$

Also, $P_3 - \text{deg}_H v_1 = n - 1 < P_3 - \text{deg}_H v_2 = 2(n - 2)$. Therefore, $H$ is a $P_3$-pairlone graph of order $n + 1$. But $H$ is not (ordinary degree) pairlone, because $\text{deg}_H v_1 = \text{deg}_H v_2 = 2$, and $H$ has two pairs of vertices of the same degree. \hfill \Box
Theorem 2.5. For any \( n \geq 6 \) there is at least one \( P_3 \)-irregular graph of order \( n \).

Proof. Let \( PL_n \) be the unique connected pairlone graph of order \( n \). This time we construct \( H \) from \( PL_n \) by adding vertex \( v_0 \) to \( V(G) \) and joining \( v_0 \) to the vertex \( v_{\mu+1} \), where \( \mu = \lfloor n/2 \rfloor \). (See Figure 8, for \( n = 5, 6, 7 \).) Now, note that \( v_0 \) is only adjacent to \( v_{\mu+1} \), that has degree \( \mu = \lfloor n/2 \rfloor \), therefore \( P_3 - \deg v_0 = \mu - 1 \). Also, this modification will not effect the \( P_3 \)-degrees of vertices \( v_1, v_2, \ldots, v_{\mu-1} \), because they are not adjacent to \( v_{\mu+1} \). Moreover, if \( v_i \), \( i \neq 0 \) is adjacent to \( v_{\mu+1} \), its \( P_3 \)-degree will be increased by one (for the new subgraph \( v_i, v_{\mu+1}, v_0 \), which is isomorphic to \( P_3 \)). Finally, \( P_3 - \deg_H v_{\mu} = P_3 - \deg_G v_{\mu} + \frac{1 + (-1)^n}{2} \), because, the parity of \( n \) will determine whether the two vertices \( v_{\mu} \) and \( v_{\mu+1} \) are adjacent or not. In summary,

\[
P_3 - \deg_H v_i = \begin{cases} 
\lfloor n/2 \rfloor - 1 & \text{for } i = 0 \\
 i(n-2) & \text{for } 0 < i < \lfloor n/2 \rfloor \\
\lfloor n/2 \rfloor(n-2) + \frac{1 + (-1)^n}{2} & \text{for } i = \lfloor n/2 \rfloor \\
2 + (i-1)(n-2) + \lfloor n/2 \rfloor - i & \text{otherwise}.
\end{cases}
\]

Clearly, \( P_3 \)-degrees of \( v_0, v_1, \ldots, v_{\mu+1} \) are different, as are \( P_3 \)-degrees of \( v_{\mu+2}, \ldots, v_n \). It only remains to show that

\[
P_3 - \deg_H v_{\mu+1} < P_3 - \deg_H v_{\mu+2},
\]

or \( |n/2|(n-2) + |n/2| < 1 + P_3 - \deg_G v_{\mu+2} \).

By (3), this is equivalent to \( |n/2|(n-2) + |n/2| < (\lfloor n/2 \rfloor + 1)(n-2) \), or \( |n/2| < n - 2 \), which is correct, since \( n \geq 6 \).

Next, we consider when \( n \leq 5 \). For \( n = 3 \), there are only two connected graphs of order three, namely \( P_3 \) and \( K_3 \). These two graphs are neither \( P_3 \)-pairlone nor \( P_3 \)-irregular; in fact, both are \( P_3 \)-regular. For \( n = 4 \), the unique connected pairlone graph of order four is the only \( P_3 \)-pairlone
graph, and there is no $P_3$-irregular graph of order four. Also, it can easily be verified that there is no $P_3$-irregular graph of order five. □

References