2.4 Permutations of Multisets

Recall: General Counting Problems

1. ordered arrangements or ordered selections
2. unordered arrangements or unordered selections

With each of (1) and (2), might allow

- No repetition
- Restricted repetition
- Unlimited repetition

Recall: Members of a set are distinct – either an element belongs or does not belong

Now: A multiset allows repeated elements (Example: \( M = \{3 \cdot a, 1 \cdot b, 2 \cdot c, 4 \cdot d\} \))

Remark: If every repetition number is 1, then the multiset is just a set.

Note: It is possible for repetition numbers to be infinite.

Example: Find six distinct 4-permutations of the multiset \( S = \{2 \cdot a, 1 \cdot b, \infty \cdot c\} \).

Example: How many 4-permutations of \( S = \{\infty \cdot a, \infty \cdot b, \infty \cdot c\} \) are there?

Question: How many \( r \)-permutations of a multiset \( S \) are there?

Easy Case – when all repetition numbers are greater than or equal to \( r \) (all repetition numbers are \( \infty \))

THEOREM Let \( S \) be a multiset with \( k \) different types of objects, where each has an infinite repetition number. Then the number of \( r \)-permutations of \( S \) is

PROOF For each \( i = 1, 2, \ldots, r \), the \( i \)th task is to

There are outcomes for each \( i = 1, 2, \ldots, r \), regardless of the outcomes of the previous tasks.

Thus, by the principle, there are \( r \)-permutations of \( S \).
Example: How many 6-permutations of $S = \{6 \cdot a, \infty \cdot b, \infty \cdot c\}$ are there?

Example: How many permutations of $S = \{2 \cdot a, 1 \cdot b, 3 \cdot c\}$ are there?

**THEOREM** Let $S$ be a multiset with objects of $k$ different types with repetition numbers, where the size of $S$ is $n = \text{...}$

Then the number of

**PROOF** Let $S = \text{...}$

Remark: If $r < n$, then
Example: How many permutations of $S = \{2 \cdot a, 1 \cdot b, 3 \cdot c\}$ are there?

One way – by the theorem

Another way – use $S' = \ldots$

8-by-8 Chessboard

Two rooks attack each other if and only if they are in

So nonattacking rooks must be in

How many nonattacking rooks may be placed on an 8-by-8 chessboard?

Example: How many ways can we place 8 nonattacking rooks on the 8-by-8 board?

Note: We have been assuming the rooks are indistinguishable (they are all identical).

Example: How many ways can we place 8 distinctly colored nonattacking rooks on the 8-by-8 board?

Example: If there are 3 red, 3 blue, and 2 green rooks, in how many ways can we place them so that they are nonattacking?
Example: How many 8-permuations of \( S = \{3 \cdot a, 2 \cdot b, 4 \cdot c\} \) are there?

Another variation of the theorem from today (see the example on page 48):

THEOREM     Let \( n \) be a positive integer and let \( k \) be positive integers with \( \sum x_k = n \). The number of ways to partition a set of \( n \) objects into \( k \) boxes in which Box 1 contains \( x_1 \) objects, Box 2 contains \( x_2 \) objects, …, Box \( k \) contains \( x_k \) objects equals

If the boxes are NOT \( \), and \( \), then the number of partitions equals