11.2 Eulerian Trails

Königsberg bridge problem

Can the townspeople take a tour of the city by traversing each bridge exactly once and end where they started?

(Euler solved this problem in 1736)

Recall:
A walk is a sequence of edges, where two consecutive edges share a common vertex.

NOTE that in a general graph \( G \), each edge has multiplicity that may be greater than 1. We do not regard an edge as repeated in a walk if the number of times it occurs in the walk does not exceed its multiplicity. An edge is repeated only if the number of times it occurs in the walk is greater than the number of “copies” available in \( G \).

A trail is a walk that has distinct edges. (Thus, in a trail the number of times an edge occurs cannot exceed its multiplicity.)

Now: A trail in a general graph \( G \) is an \( n \)-trail if it contains \( n \) of \( G \).

THEOREM (Euler) A connected general graph \( G \) has a closed Eulerian trail if and only if

Lemma Let \( G = (V, E) \) be a general graph, every vertex of which has \( n \). Then each edge of \( G \) belongs to a \( n \) and hence to a \( n \).

Proof Let \( e \) be an edge of \( G \). We use an algorithm to construct a closed trail containing \( e \).
The algorithm constructs a set $W$ of vertices and a multiset $F$ of edges (number of times an edge can be in $F$ is no more than its multiplicity in $G$).

**Algorithm**

1. Put $i = 1$
2. Put $W = \{ \}$
3. Put $F = \{ \}$
4. While not in $F$
   (a) Locate an edge not in $F$
   (b) Put in $W$ (if is already in $W$,
   (c) Put in $F$
   (d) Increase $i$ by 1.

Example:

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i  x_i  \alpha_i  W  F
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THEOREM (Euler) A connected general graph $G$ has a closed Eulerian trail if and only if

Proof First, let $G$ be a connected general graph. Suppose that $G$ has a closed Eulerian trail. Let $u$ be a vertex of $G$. Then either $u$ is the first (and also the last) vertex on the trail or $u$ is not the initial vertex of the trail.

If $u$ is NOT the initial vertex on the trail, then

If $u$ is the initial vertex of the trail, then

For the converse, we assume that

Let $G_1 = (V, E_1)$ be the general graph $G$. Choose an $u$ of $G_1$. By our lemma, there is a

Let $G_2 = (V, E_2)$ be the general graph $G_1$ with the edges of $E_2$ contains the edges of $G_1$ that are not on $E_2$. Notice that the vertices of $G_2$ have

If $E_2$ contains an edge, then because $G_1$ is $E_2$ has an edge that is incident to a vertex of $G_2$.

Since the edge is on a $E_2$. Then we have our closed Eulerian trail of $G$.

Continue in this manner, defining

Continue until all edges of are on a $E_2$. Then we have our closed Eulerian trail of $G$. 
THEOREM Let $G$ be a connected general graph. Then $G$ has an edge if and only if there are exactly two vertices $u$ and $v$ of odd degree. Every edge in $G$ joins $u$ and $v$.

Proof Let $G$ be a connected general graph and assume that $G$ has an edge, say that the trail starts at $u$ and ends at $v$. Then
\[
\deg(u) = \deg(v) = \deg(w) =
\]
For the converse, let $u$ and $v$ denote the two vertices of $G$ having odd degree. Then let $G'$ be the general graph obtained from $G$ by

THEOREM Let $G$ be a connected general graph and let $m \geq 0$ denote the number of vertices of $G$ with odd degree. Then the edges of $G$ can be partitioned in $m/2$ (and no fewer) open trails.

Chinese postman problem: Let $G$ be a connected general graph. Find a path of shortest length which uses each edge of $G$ at least once.

THEOREM Let $G$ be a connected general graph having $K$ edges. Then there is a trail in $G$ of length $2K$ in which the number of times an edge is used equals twice its multiplicity.

Proof Let $G'$ be the general graph obtained from $G$ by adding the

Then $G'$ is a graph with $m$ edges. Also every vertex of $G'$ has a trail. Thus $G'$ has a trail. This trail in $G'$ corresponds to a

Example: To see that this is ‘best possible’ in a certain sense, consider the following graph.