6.6 Improper Integrals

Recall: \[ \int_a^b f(x) \, dx = F(b) - F(a) \], where \( F'(x) = f(x) \)

This works when

NOW: Consider when either

1. 

or

2. 

When either of these occur, the integral is called an improper integral.

In case of an improper integral of type 1: \( \int_a^\infty f(x) \, dx \)

If the limit exists, then the improper integral exists and equals the limit, and we say that the integral \( \int_a^\infty f(x) \, dx \) converges to \( \lim_{b \to \infty} \int_a^b f(x) \, dx \) or that \( \int_a^\infty f(x) \, dx \) is convergent.

If the limit does not exist, then the improper integral does not exist, and we say that the integral \( \int_a^\infty f(x) \, dx \) diverges or that it is divergent.

Another case of an improper integral of type 1: \( \int_{-\infty}^b f(x) \, dx \)

If both \( \int_a^\infty f(x) \, dx \) and \( \int_{-\infty}^a f(x) \, dx \) exist, then we define \( \int_{-\infty}^\infty f(x) \, dx = \)

**EXAMPLES**

1. \( \int_0^\infty e^{-x} \, dx = \)

2. \( \int_0^\infty \frac{1}{1 + x^2} \, dx = \)
3. Sketch the region determined by the set \( S = \{(x, y) \mid x \geq 1, \ 0 \leq y \leq 1/x\} \) and find its area (if the area is finite).

4. For what values of \( p \) is the integral \( \int_1^{\infty} \frac{1}{x^p} \, dx \) convergent?

In case of an improper integral of type 2:

(a) If \( f \) is discontinuous at \( b \), then define 
\[
\int_a^b f(x) \, dx = \lim_{t \to b^-} \int_a^t f(x) \, dx \quad \text{if the limit exists.}
\]

(b) If \( f \) is discontinuous at \( a \), then define 
\[
\int_a^b f(x) \, dx = \lim_{t \to a^+} \int_t^b f(x) \, dx \quad \text{if the limit exists.}
\]

(c) If \( f \) is discontinuous at \( c \), where \( a < c < b \), then define 
\[
\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx
\]
EXAMPLES

1. \[ \int_0^1 \frac{1}{\sqrt{1-x^3}} \, dx \]

2. \[ \int_0^1 \frac{1}{x} \, dx \]

3. \[ \int_{-1}^1 \frac{1}{x^2} \, dx \]

Comparison Theorem Let \( f(x) \) and \( g(x) \) be continuous functions with \( f(x) \geq g(x) \geq 0 \) for \( x \geq a \). Then

(i) If \( \int_a^\infty f(x) \, dx \) is convergent, then \( \int_a^\infty g(x) \, dx \)

(ii) If \( \int_a^\infty g(x) \, dx \) is divergent, then \( \int_a^\infty f(x) \, dx \)

EXAMPLE Determine whether the integral \( \int_0^\infty \frac{\sin^2 x}{1+x^2} \, dx \) converges or diverges.