5.2 The definite integral

Let \( f(x) \) be defined on \([a, b]\). (Not necessary that \( f(x) \geq 0 \))

Divide \([a, b]\) into \( n \) equal length subintervals. (Each of length \( \Delta x = \frac{b-a}{n} \))

Choose \( x_i^* \) in the \( i \)th subinterval \([x_{i-1}, x_i]\).
(Usually we choose \( x_i^* \) to be the right endpoint.)
(Midpoint Rule is when \( x_i^* \) is chosen to be the midpoint of the \( i \)th subinterval.)

Compute the Riemann Sum \( \sum_{i=1}^{n} f(x_i^*) \Delta x \)

Define \( L = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x \) to be the integral of \( f(x) \) on \([a, b]\).

If the limit exists, we say \( f \) is integrable.

NOTATION: \( \int_{a}^{b} f(x) dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x \)

\( \int_{a}^{b} f(x) dx \) is called the definite integral of \( f(x) \) on \([a, b]\).

\( a \) and \( b \) are called the lower and upper limits of integration.

THEOREM If \( f \) is continuous, then the limit exists and so \( f \) is integrable.

EXAMPLE Express \( \lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{1 + (x_i^*)^2} \Delta x \) as an integral on the interval \([0, 1]\).

Define

1. \( \int_{a}^{a} f(x) dx = \)

2. If \( a < b \), then \( \int_{b}^{a} f(x) dx = \)
EXAMPLE  
Find $\int_{1}^{2} x^2 \, dx$ by the limit definition.

Properties

1. $\int_{a}^{b} cf(x) \, dx = c \int_{a}^{b} f(x) \, dx$

2. $\int_{a}^{b} (f(x) + g(x)) \, dx = \int_{a}^{b} f(x) \, dx + \int_{a}^{b} g(x) \, dx$

3. $\int_{a}^{b} f(x) \, dx = \int_{a}^{c} f(x) \, dx + \int_{c}^{b} f(x) \, dx$ for $a < c < b$

4. $\int_{a}^{b} c \, dx = c(b - a)$

EXAMPLES

1. Given $\int_{-1}^{3} x^3 \, dx = 20$, find $\int_{-1}^{3} (x^3 - 2) \, dx$

2. Use area to find $\int_{0}^{6} (5 - x) \, dx$