4.3 Derivatives and shapes of graphs
4.4 Curve sketching

A function \( f \) is **increasing** on an interval \( I \) if whenever \( x_1 \) and \( x_2 \) are in \( I \) and \( x_1 < x_2 \), then \( f(x_1) < f(x_2) \).

A function \( f \) is **decreasing** on an interval \( I \) if whenever \( x_1 \) and \( x_2 \) are in \( I \) and \( x_1 < x_2 \), then \( f(x_1) > f(x_2) \).

**Test for a function to be increasing or decreasing:**

Let \( f \) be continuous on \( [a, b] \) and differentiable on \( (a, b) \).

1. If \( f'(x) > 0 \) for every \( x \) in \( (a, b) \), then \( f \) is increasing on \( (a, b) \).

2. If \( f'(x) < 0 \) for every \( x \) in \( (a, b) \), then \( f \) is decreasing on \( (a, b) \).

**EXAMPLES** Find the intervals on which each function is increasing or decreasing.

1. \( f(x) = x^3 - 3x^2 + 2 \)

2. \( f(x) = x^3 - 3x^2 + 3x + 5 \)
RECALL  Local minimum values and local maximum values occur at critical numbers:

Local max

Local min

Remark:
1. If \( f'(x) \) changes sign as you pass through a critical number, then \( f \) has a local min or local max at this critical number
2. The way it changes sign tells you which type (local min or local max).

FIRST DERIVATIVE TEST  (to find local minimum and local maximum values)

(1) Find critical points
(2) Determine whether (and how) \( f'(x) \) changes sign as you pass through each critical number.

EXAMPLES  Find the intervals where each function is increasing and decreasing. Find the local minimum and local maximum values. Use this information to sketch the graph.

1. \( f(x) = \frac{1}{3} x^3 - \frac{1}{2} x^2 + 1 \)
2. \( f(x) = x^3 + 3x^2 + 3x - 1 \)

3. \( f(x) = x^{4/3} + 32x^{1/3} \)

Let \( f \) be differentiable on \((a, b)\).

If \( f' \) is increasing on \((a, b)\), then \( f \) is **concave up** on \((a, b)\).

If \( f' \) is decreasing on \((a, b)\), then \( f \) is **concave down** on \((a, b)\).

Points where \( f \) switches back and forth from concave up to concave down are called **points of inflection**.

How do we check for \( f' \) increasing or decreasing?

\[
\begin{align*}
  f''(x) > 0 & \text{ implies } f' \text{ is increasing } \implies f \text{ is concave up} \\
  f''(x) < 0 & \text{ implies } f' \text{ is decreasing } \implies f \text{ is concave down}
\end{align*}
\]
So if \( f'' (x) \) changes sign as you pass through \( x = c \), then \( c \) is a point of inflection.
The way \( f'' (x) \) changes sign tells you whether \( f \) goes from concave up to concave down or vice versa.

**SECOND DERIVATIVE TEST** (to find local minima and local maxima)

1. If \( f \) has a critical point \( c \) such that \( f'' (c) \) exists and \( f'' (c) > 0 \), then \( f \) has a local minimum at \( c \) (\( f(c) \) is a local min)
2. If \( f \) has a critical point \( c \) such that \( f'' (c) \) exists and \( f'' (c) < 0 \), then \( f \) has a local maximum at \( c \) (\( f(c) \) is a local max)

NOTE: The Second Derivative Test says NOTHING for when \( f'' (c) = 0 \) nor for when \( f'' (c) \) does not exist.

**EXAMPLE** For the function \( f(x) = 6x^5 - 10x^3 \), find critical points, local minimum and local maximum values, points of inflection, and intervals on which the graph is increasing, decreasing, concave up, and concave down. Use this information to sketch the graph.