1. (4 pts.) Find the domain and range of the function \( f(x) = \sqrt{x^2 - 4} \).

\[
\begin{align*}
  &x^2 - 4 \geq 0 \\
  &x^2 - 4 \geq 0 \\
  &x - 2 \leq 0 \quad \text{or} \quad x + 2 \leq 0 \\
  &x \leq 2 \quad \text{or} \quad x \geq -2
\end{align*}
\]

Domain: \( (-\infty, -2] \cup [2, \infty) \)

Range of \( y = x^2 - 4 \) in \( (-\infty, -2] \cup [2, \infty) \) and \( y \geq 0 \), so range of \( f(x) = \sqrt{x^2 - 4} \) is \( [0, \infty) \).

2. (30 pts.) Evaluate the limit, if it exists. If the limit does not exist, explain why.

(a) \[ \lim_{x \to 0} \frac{\sin 3x}{x} = \lim_{x \to 0} \frac{3 \cdot \sin 3x}{3x} = 3 \lim_{x \to 0} \frac{\sin 3x}{3x} = 3 \cdot 1 = 3 \]

(b) \[ \lim_{x \to 3} \frac{x^2 + 2x - 3}{x^2 + x - 6} = \lim_{x \to 3} \frac{(x + 3)(x - 1)}{(x + 3)(x - 2)} = \lim_{x \to 3} \frac{x - 1}{x - 2} = \frac{-1}{-5} = \frac{4}{5} \]

(c) \[ \lim_{x \to 3} \frac{|3 - x|}{x - 3} \text{ does not exist because} \]

\[ \lim_{x \to 3^+} \frac{3 - x}{x - 3} = \lim_{x \to 3^+} \frac{-2}{x - 3} = \lim_{x \to 3^+} \frac{-2}{x - 3} = -2 \]

\[ \lim_{x \to 3^-} \frac{3 - x}{x - 3} = \lim_{x \to 3^-} \frac{-2}{x - 3} = \lim_{x \to 3^-} \frac{-2}{x - 3} = -2 \]

(d) \[ \lim_{x \to \infty} \left( \frac{3x + 5x^2 - 2x^4}{5x^4 - 3x^2 + 2} \right) = \lim_{x \to \infty} \frac{3x + 5x^2}{x^4} \cdot \frac{x^{-4}}{x^{-4}} = \lim_{x \to \infty} \frac{3}{x^3} + \frac{5}{x^2} - 2 = \frac{-2}{5} \]

(e) \[ \lim_{x \to 2} \frac{x^2 + 2 - 2}{x - 2} = \lim_{x \to 2} \frac{(x^2 + 2) - 2}{x - 2} = \lim_{x \to 2} \frac{x^2 + 2}{x - 2} \cdot \frac{1}{(x + 2)(x - 2)} = \lim_{x \to 2^+} \frac{x - 2}{x + 2} = \frac{1}{4} \]
3. (6 pts.) Determine (with explanation) the infinite limit \( \lim_{x \to -\infty} \frac{x-5}{x-2} (x-2)(x+3) \)

\[
\lim_{x \to -2^-} \frac{x-5}{x+3} = -3 \quad \text{and} \quad \lim_{x \to 2^+} (x-2) (x+3) = 0
\]

So \( \lim_{x \to -2^-} \frac{x-5}{x+3} \) does not exist.

As \( x \to 2^- \), we see \( -\infty \to -3 \) \( (neg) \) \( \quad \) \( 0 \) \( (neg) \) \( \to \infty \).

So \( \lim_{x \to 2^-} (x-2) (x+3) = \infty \).

4. (6 pts.) Use the Squeeze Theorem to find \( \lim_{x \to 0} \frac{x^4 \sin \left( \frac{1}{x} \right)}{x} \).

For \( x \to 0 \), note that

\( -1 \leq \sin \frac{1}{x} \leq 1 \)

Multiplying through by \( x \),

we obtain \( -x \leq x^4 \sin \frac{1}{x} \leq x \).

Notice \( \lim_{x \to 0} -x = 0 = \lim_{x \to 0} x \). Thus by the Squeeze Theorem,

\( \lim_{x \to 0} x^4 \sin \frac{1}{x} = 0 \).

5. (12 pts.) Consider the function \( f(x) = \begin{cases} \frac{2x-1}{x} & x < 3 \\ x^2 + 1 & 3 \leq x < 4 \\ 3x + 5 & x \geq 4 \end{cases} \).

(a) Find (with explanation) the numbers at which \( f \) is discontinuous.

\( x = 3 \)

\( f(3) = \frac{2 \cdot 3 - 1}{3} + 1 = 10 \)

\( \lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \frac{2x-1}{x} = \frac{5}{3} \)

\( \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (x^2 + 1) = 10 \)

So \( \lim_{x \to 3} f(x) \) does not exist.

\( f \) is discontinuous at \( x = 3 \).

(b) At which of these points is \( f \) continuous from the right, from the left, or neither?

Since \( f(3) = 10 = \lim_{x \to 3^-} f(x) \), we see \( f \) is continuous from the right at \( x = 3 \).

(c) Sketch the graph of \( f \).
6. (6 pts.) Use the Intermediate Value Theorem to show that there is a root of the equation $2x^3 + x^2 + 2 = 0$ in the interval $(-2, -1)$.

Let $f(x) = 2x^3 + x^2 + 2$, which is continuous on $[-2, -1]$. Also, $f(-2) = -16 + 4 + 2 = -10 < 0$ and $f(-1) = -3 + 1 + 2 = 1 > 0$. Since $f(-2) < 0$ and $f(-1) > 0$, there is a number $c$ in $(-2, -1)$ with $f(c) = 0$. This $c$ is the desired root.

7. (8 pts.) Calculate the appropriate limits to find the horizontal and vertical asymptotes of the curve $y = \frac{3x + 1}{2x - 8}$. Then graph the curve using this information.

Vertical asymptotes:
1. $\lim_{x \to 4^{-}} \frac{3x + 1}{2x - 8} = -\infty$
2. $\lim_{x \to 4^{+}} \frac{3x + 1}{2x - 8} = \infty$

$x = 4$ is a vertical asymptote.

Horizontal asymptotes:
1. $\lim_{x \to \infty} \frac{3x + 1}{2x - 8} = \frac{3}{2}$
2. $\lim_{x \to -\infty} \frac{3x + 1}{2x - 8} = \frac{3}{2}$

$y = \frac{3}{2}$ is a horizontal asymptote.

8. (10 pts.) Find an equation of the tangent line to the graph of $y = \frac{3x}{x + 1}$ at the point $(-5, -2)$.

$$M_{tan} = \lim_{h \to 0} \frac{f(-5+h) - f(-5)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{3(-5+h)}{-5+h+1} - (-2)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{8-h}{h-4} + 2}{h}$$

$$= \lim_{h \to 0} \frac{8-h+2(h-4)}{h(h-4)}$$

$$= \lim_{h \to 0} \frac{8-8h+2h-8}{h(h-4)}$$

Equation of tangent:
$$y + 2 = -\frac{1}{4}(x + 5)$$
9. (8 pts.) A particle moves along a straight line with equation of motion \( s = f(t) = 1 - t^2 \), where \( s \) is measured in meters and \( t \) in seconds. Find the velocity when \( t = 3 \) seconds.

\[
\begin{align*}
v'(3) &= \lim_{{h \to 0}} \frac{f(3+h) - f(3)}{h} \\
&= \lim_{{h \to 0}} \frac{1 - (3+h)^2 - (-8)}{h} \\
&= \lim_{{h \to 0}} \frac{1 - (9 + 6h + h^2) + 8}{h} \\
&= \lim_{{h \to 0}} \frac{9 - 9 - 6h - h^2}{h} \\
&= \lim_{{h \to 0}} \frac{h(-6-h)}{h} \\
&= \lim_{{h \to 0}} (-6-h) = -6 \text{ m/s}
\end{align*}
\]

10. (10 pts.) Find the derivative of \( f(x) = \sqrt{x-2} \) using the definition of derivative. State the domain of the function and the domain of its derivative.

\[
f'(x) = \lim_{{h \to 0}} \frac{f(x+h) - f(x)}{h}
\]

\[
= \lim_{{h \to 0}} \frac{\sqrt{x+h-2} - \sqrt{x-2}}{h} \\
= \lim_{{h \to 0}} \frac{(\sqrt{x+h-2} - \sqrt{x-2})(\sqrt{x+h-2} + \sqrt{x-2})}{h(\sqrt{x+h-2} + \sqrt{x-2})} \\
= \lim_{{h \to 0}} \frac{x+h-2 - (x-2)}{h(\sqrt{x+h-2} + \sqrt{x-2})} \\
= \lim_{{h \to 0}} \frac{1}{\sqrt{x+h-2} + \sqrt{x-2}} \\
= \frac{1}{2\sqrt{x-2}}
\]

Domain of \( f(x) = \sqrt{x-2} \) is \([2, \infty)\)