A LIMIT PROBLEM

Limits of functions of more than one variables are not easy. The following problem illustrates that.
Find
\[ \lim_{(x,y) \to (0,0)} \frac{x^2}{x + y} \]

\(x^2\) has higher power than \(x + y\). One would expect the limit to be 0.

The domain of the function is given by \(y \neq -x\), i.e. all points of \(\mathbb{R}^2\) except the points on the straight line \(y = -x\).

Hence, on every straight line containing \(O(0,0)\),
\[ \lim_{(x,y) \to (0,0)} \frac{x^2}{x + y} = 0 \]

Is \(\lim_{(x,y) \to (0,0)} \frac{x^2}{x + y} = 0\)?

Let’s try polar coordinates as in the textbook. \(x = r \cos \theta\) and \(y = r \sin \theta\).
\[ \lim_{(x,y) \to (0,0)} \frac{x^2}{x + y} = \lim_{r \to 0} \frac{r^2 \cos^2 \theta}{r \cos \theta + r \sin \theta} = \lim_{r \to 0} \frac{r \cos^2 \theta}{\cos \theta + \sin \theta} = 0?? \]

Again, \(\cos \theta + \sin \theta \neq 0\) because \(y \neq -x\).
So, have you been convinced that
\[ \lim_{(x,y) \to (0,0)} \frac{x^2}{x + y} = 0 ?? \]

No! No! No!
\(\lim_{(x,y) \to (0,0)} \frac{x^2}{x + y} = 0\) is false!!!
Indeed, on the line \(y = x^2 - x\) (containing \(O(0,0)\)),
\[ \lim_{y = x^2 - x} \frac{x^2}{x + y} = \lim_{x \to 0} \frac{x^2}{x + x^2 - x} = \lim_{x \to 0} \frac{x^2}{x^2} = 1 \neq 0 \]

Finally, we conclude that \(\lim_{(x,y) \to (0,0)} \frac{x^2}{x + y}\) doesn’t exist.