ALGEBRAIC FORMULAS FOR A BIQUADRATIC EQUATION IN ALL POSSIBLE CASES

Consider the general biquadratic equation:

\[ \frac{1}{4}x^4 + px^2 + q = 0 \]

It is erroneously believed that the substitution \( x^2 = y \) could lead to a general algebraic formula. The following is a set of 4 formulas that produce this result in all possible cases instead.

Case 1: \( q > p^2 \)
The equation has 4 imaginary roots and they are given by

\[ x_{1,2,3,4} = \pm \sqrt{\sqrt{q} - p} \pm i\sqrt{\sqrt{q} + p} \]

Case 2: \( p \leq 0 \leq q \leq p^2 \)
The equation has 4 real roots and they are given by

\[ x_{1,2,3,4} = \pm \sqrt{-p + \sqrt{q}} \pm \sqrt{-p - \sqrt{q}} \]

Case 3: \( p > 0 \) and \( 0 \leq q \leq p^2 \)
The equation has 4 purely imaginary roots and they are given by

\[ x_{1,2,3,4} = i(\pm \sqrt{p + \sqrt{q}} \pm \sqrt{p - \sqrt{q}}) \]

Case 4: \( q < 0 \)
The equation has 2 real roots and 2 purely imaginary roots given by

\[ x_{1,2} = \pm \sqrt{2\sqrt{p^2 - q} - 2p} \]

\[ x_{3,4} = \pm i\sqrt{2\sqrt{p^2 - q} + 2p} \]

The discriminant of the biquadratic equation is \( D = -q^2 + p^2q = q(p^2 - q) \).