Stalemates in Bilateral Bargaining

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Abstract

In this paper, we develop bilateral bargaining models in which bargainers have an outside option that provide conditions for a stalemate to occur. We provide game theoretic motivations for bargainer behavior by developing their respective decisions as a result of utility maximization problems. The respective utility maximization problems are impractical for an analysis of the conditions for stalemates to occur. Thus, a system of difference equations is used to describe bargainer counteroffers. The descriptive model provides numerous conditions for stalemates to occur. It also provides a detailed description of the equilibrium in other cases.

1. Introduction

Since the work of Nash [27, 28] on the classic two-person bargaining problem, much work has been done to expand the range and usefulness of bargaining models. Shortly thereafter Schelling [33] wrote that bargaining “covers wage negotiations, tariff negotiations, competition where competitors are few, settlements out of court, and the real estate agent and his customer.”

A significant portion of bargaining literature, including the present paper, dedicates itself to situations akin to that between the real estate agent and his customer.

Rubinstein [30] constructs a commonly used alternating offers bargaining game where two players must reach an agreement over how to divide a pie between themselves. Rubinstein shows that with costly delay there exists a unique perfect equilibrium where both players agree within the first period. However, according to experimental results, as we later discuss, instant agreement seldom occurs in bargaining. For Rubinstein’s game, this arises out of the assumption that for any division of the pie at a given time, there exists an equally preferred division of the pie at an earlier time. We modify this alternating offers bargaining game to present a game that is better aligned with experimental results.

Repeated bargaining games have been widely studied since Rubinstein [30]. Fudenberg and Tirole [15] consider a two-period bargaining game between the buyer and the seller of an indivisible good. Fudenberg and Tirole analyze the role played by incomplete information and determine the perfect Bayesian equilibrium. Other work on bilateral bargaining similar to the aforementioned work considers various implications including in markets, with strategic delays, and with uncertainty, e.g. Rubinstein and Wolinsky [31], Admati and Perry [1], Sákovics [32], Thanassoulis [34], Feinberg and Skrzypacz [13], and Dutta [10]. A summary of earlier models is provided by Binmore et al. [4].

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The role played by outside options in bargaining is of particular interest in the present paper. Muthoo [26] posits an alternating offers bargaining game in which one player has the option of searching for an outside option. Muthoo finds that the search process is equivalent to the player having a single given outside option at the outset. Other work suggests that outside options lead to inefficiencies such as delays or lost surplus. For example, Fuchs and Skrzypacz [14] demonstrate that arrival of new traders or information creates delay and forces the seller to lower their price to the value of their outside option. Hwang and Li [19] incorporate the role of incomplete information and consider a bilateral bargain where the buyer has an outside option that arrives either publicly or privately. Hwang and Li find that public arrivals benefit the buyer but their value should not be disclosed. Other papers that examine outside options in bargaining include Lee and Liu [23], Chang [8], Board and Pycia [6], and Hwang [18]. The present paper differs by demonstrating that outside options prompt bargainers to commit to prior offers and thus lead to stalemates.

Unfortunately, many models for alternating offers bargaining games, e.g. Rubinstein [30] and Muthoo [26], contradict pragmatic and experimental perspectives on bargaining by making restrictive assumptions on player preferences. Elsewhere, e.g. Fudenberg and Tirole [15], the game’s scope is limited by considering two-stage games or forcing players to present an ultimatum after a fixed amount of turns. While these approaches provide economic insights, they are not descriptive of reality.

The use of backward induction to derive equilibria, e.g. subgame perfect equilibria or Bayesian equilibria, is not in accordance with typical human behavior in bargaining. Early results on bargaining experiments are summarized by Roth [29] who concludes that theoretical subgame perfect equilibrium are seldom achieved in bargaining experiments with discount factors and that fairness often plays a role in behavior (see Weg et al. [35] and Binmore et al. [3]). Weg et al. suggest that some people are driven by equity and equality while bargaining. Meanwhile, evidence suggests that people use backward induction while playing certain games, e.g. the race game in Gneezy et al. [16] and Brosig-Koch et al. [7]. However, Johnson et al. [21] and Binmore et al. [5] provide evidence that backward induction is not utilized by players in bargaining games. On the basis of this evidence, we abandon backward induction for the bargaining game presented in this paper.

We develop a modified form of the bargaining game presented by Rubinstein [30] with costly delay. We use price haggling between a single buyer and a single seller as a narrative device throughout. Our bargaining game has players driven by short-term interests that do not use backward induction as suggested by aforementioned experimental evidence. This allows for a model that is more descriptive of reality. This approach is not entirely novel and bears similarity to the model of Fuchs and Skrzypacz [14].

Buyer and seller present initial offers and have exogenous outside options whose existence is common knowledge but whose value is known only to the respective player. Players may accept their opponent’s previous counteroffer, propose a new counteroffer, or exit and receive their outside option. The expected utility is developed for buyers and presented similarly for sellers. The complexity of the respective utility maximization problems makes it unmanageable for a practical analysis of the conditions under which stalemates occur. A system of difference equations is used to provide a tractable framework in which these conditions are examined.

The analysis of stalemates is an active component of research in bargaining. Schelling [33] considers disagreements in bargains as the result of commitment. Later, Crawford [9] considers a commitment game in a bargaining context to expand
work by Schelling. To the best of our knowledge, little else has been done on stalemates that is relevant to our bargaining model until recently. Ellingsen and Miettinen [11, 12] find that bargainers commit to prior offers to force their opponents to concede and thus obtain a greater surplus. Li [25] finds further conditions for an impasse to arise in a bargaining game.

Our model coalesces the yet disjoint research on bargaining with outside options and bargaining stalemates. We present a game theoretic model where bargainers commit based on the strength of their outside option. A more convenient setting is provided by modeling counteroffers using a general system of difference equations. Solving this system provides conditions for the sequence of counteroffers under which stalemates arise in the bilateral bargaining game.

Our results are aligned with those of Li [25], even if our models are entirely different. Li utilizes backward induction to find subgame perfect equilibrium when bargainers are unable to successfully commit and shows that stalemates occur when the probability for successful commitments is high. This approach is at odds with the findings of experimental evidence as previously discussed. We establish limited lookahead utility maximization problems for the bargainers and use it to motivate a system of difference equations to model bargainer behavior. We solve the system of difference equations to demonstrate that intransigent bargainers that grow increasingly committed to their previous counteroffers can create stalemates and force both players to gain less than they would from a successful bargain. Bargainers are shown to commit in the presence of strong outside options. The present paper depicts a tangible context in which stalemates occur and more specific conditions for stalemates to occur.

The paper is organized as follows: Section 2 presents the game theoretic motivations; Section 3 presents a preliminary model that utilizes a simple system of difference equations to describe bargainer counteroffers; Section 4 presents a general framework for bargainer counteroffers as a system of difference equations; Section 5 presents conditions for stalemates; Section 6 presents several illustratory examples; Section 6 concludes.

2. Motivations

Consider a bargaining game in which two players, named the buyer and the seller, propose competing counteroffers over the price of a good. The price is normalized to the unit interval \([0, 1]\) so that this game is akin to the Rubinstein alternating offers bargaining game. The game is initialized by the seller proposing an offer of 1 and the buyer proposing a counteroffer of 0. After initialization, the seller takes the first turn. Each turn of the seller (buyer) is then followed by a turn of the buyer (seller).

The buyer and the seller have outside options \(c_b\) and \(c_a\) in \((0, 1)\) that are given prior to the start of the bargaining process. They have entered the bargain in an attempt to obtain a bargain preferred to the outside option. While each bargainer knows they existence of the other’s outside option, they do not know the strength. It is assumed that the value of the outside options does not diminish as the bargaining process continues.

On each player’s turn, they may: (i) exit the bargain and take the outside option, (ii) accept the opponent’s last offer, or (iii) propose a counteroffer. We assume that the value of counteroffers diminishes with time using discounting factors for the buyer and seller \(\delta_b, \delta_s \in (0, 1]\). If either discounting factor is equal to 1, then the corresponding bargainer has costless delay. This is demonstrated in Figure 1 together with buyer and seller payoffs for exiting or accepting counteroffers.
The actions of buyers and sellers are not dictated under the assumption that they utilize backward induction to tend toward an equilibrium. Instead, each player acts on limited lookahead. They base their decision on the expected utility after their next turn, dependent on what they believe their opponent might do. The dependence on the opponent’s player is dictated according to a sequence of underlying conditional probability distributions.

We now analyze the buyer’s behavior. Conclusions for the seller’s behavior follow similarly and are summarized afterwards. We begin by considering turn $t$ of the buyer. On this turn, the buyer decides to exit and take their outside option, accept the seller’s previous counteroffer, or propose a new counteroffer. The first two options have corresponding utilities $u_{bt} = 1 - c_b$ and $u_{bt} = \delta_t (1 - s_t)$. The buyer’s expected utility is more involved if they propose a new counteroffer.

We suppose that the buyer behaves with limited lookahead, as suggested by experimental data. The buyer’s choice of $b_t$ when proposing a counteroffer depends on the maximum expected utility from accepting the counteroffer, exiting, or proposing their own counteroffer. In part, this is dependent on the buyer’s maximum expected utility for the next turn given information on the present turn. The maximum expected utility on turn $t + 1$ from the perspective of the present turn is

$$u_{t+1} = \max \left\{ P_t^b (A) \delta_{t+1} (1 - b_{t+1}) + P_t^b (C) u_{t+2} + P_t^b (E) (1 - c_b), 1 - c_b \right\}$$

(2.1)
where \( b_{t+1} \) is chosen to maximize expected utility. The recursive relationship between \( \tilde{u}_{t+1}^b \) and \( \tilde{u}_{t+2}^b \) suggests that the buyer must have information on components of \( P_{t+1}^b \) to determine \( \tilde{u}_{t+1}^b \). However, rearranging terms in the recursive relationship and suitably shifting \( t \) provides a relationship between \( \tilde{u}_{t+1}^b \) and \( \tilde{u}_{t}^b \) that relies entirely on prior information. Then the definition in (2.1) can be rewritten as

\[
\tilde{u}_{t+1}^b = \max \left\{ \frac{\tilde{u}_t^b - P_{t-1}^b (A) \delta_b^t (1 - b_t) - P_{t-1}^b (E) (1 - c_b)}{P_{t-1}^b (C)}, 1 - c_b \right\}.
\]  

(2.2)

The buyer’s expected utility for proposing a counteroffer is

\[
u_t^b = P_t^b (A) \delta_b^t (1 - b_t) + P_t^b (C) \tilde{u}_{t+1}^b + P_t^b (E) (1 - c_b)
\]

\[
= \delta_b^t \left( P_t^b (A) - \frac{P_{t-1}^b (A) P_t^b (C)}{P_{t-1}^b (C) - P_t^b (C)} (1 - b_t) + \frac{P_t^b (C)}{P_{t-1}^b (C) - P_t^b (C)} \tilde{u}_t^b + \left( P_t^b (E) - \frac{P_{t-1}^b (E) P_t^b (C)}{P_{t-1}^b (C) - P_t^b (C)} \right) (1 - c_b) \right).
\]  

(2.3)

Of course, if the buyer adjusts \( \tilde{u}_{t+1}^b \) ex post, then \( \tilde{u}_t^b = u_t^b \) in (2.3). This yields

\[
u_t^b = \delta_b^t \left( \frac{P_{t-1}^b (C) P_t^b (A) - P_{t-1}^b (A) P_t^b (C)}{P_{t-1}^b (C) - P_t^b (C)} (1 - b_t) + \left( \frac{P_{t-1}^b (C) P_t^b (E) - P_{t-1}^b (E) P_t^b (C)}{P_{t-1}^b (C) - P_t^b (C)} \right) (1 - c_b) \right).
\]  

(2.4)

where the buyer selects \( b_t \) to maximize \( \tilde{u}_t^b \) in (2.4).\footnote{The expression for \( u_t^b \) in (2.4) requires that \( P_{t-1}^b (C) \neq P_t^b (C) \) so that the probability of the seller proposing a counteroffer must change each period for (2.4) to be a valid expression for \( u_t^b \). Costly delay implies that the seller becomes less likely to propose a counteroffer so that \( P_t^b (C) \) is monotonically decreasing. It would then be practical to assume that \( P_t^b (C) \) is strictly decreasing.}

Succinctly,

\[
u_t^b = \max \left\{ \delta_b^t \left( \frac{P_{t-1}^b (C) P_t^b (A) - P_{t-1}^b (A) P_t^b (C)}{P_{t-1}^b (C) - P_t^b (C)} (1 - b_t) + \left( \frac{P_{t-1}^b (C) P_t^b (E) - P_{t-1}^b (E) P_t^b (C)}{P_{t-1}^b (C) - P_t^b (C)} \right) (1 - c_b) \right),
\right.

\[
1 - c_b, \delta_b^t (1 - s_t) \right\}.
\]  

(2.5)

Similarly, the seller has utility

\[
u_t^s = \max \left\{ \delta_s^t \left( \frac{P_{t-1}^s (C) P_t^s (A) - P_{t-1}^s (A) P_t^s (C)}{P_{t-1}^s (C) - P_t^s (C)} s_t + \left( \frac{P_{t-1}^s (C) P_t^s (E) - P_{t-1}^s (E) P_t^s (C)}{P_{t-1}^s (C) - P_t^s (C)} \right) c_s, \right.
\right.

\[
c_s, \delta_s^{t-1} b_{t-1} \right\}.
\]  

(2.6)

Returning to the utility maximization problem of the buyer, one notes that each of the probabilities in (2.5) are dependent on \( b_t \). This puts the buyer into a difficult situation. For example, faced with a self-interested seller, if the buyer decreases \( b_t \) then \( P_t^b (A) \) will likely decrease. This trend may reverse or attenuate given a generous or altruistic seller. In any case, it is clear that if \( c_b \approx 0 \) then the buyer has the incentive to make \( b_t \) smaller. As a result, strong outside options lead to buyers growing increasingly committed to previous offers. This follows similarly for the seller. In addition, if the buyer proposes a new counteroffer that is smaller than their previous counteroffer, they lower their utility. Thus, \( b_t \) will be monotonically increasing. Similarly, \( s_t \) will be monotonically decreasing. Unfortunately, this utilitarian approach is not practical for an analysis of how counteroffers create stalemates.
3. Preliminary Model

We set the utility maximization problems aside and consider how the buyer’s choice of \( b_t \) and the seller’s choice of \( s_t \) dictate the outcome of the bargain. We consider a model in which the buyer and seller propose counteroffers at some fixed proportion between previous offers. Let \( \{s_n\}_{n=0}^{+\infty} \) and \( \{b_n\}_{n=0}^{+\infty} \) represent the seller’s and buyer’s offers, if counteroffers were planned ad infinitum. In this simple model they have the relationship

\[
\begin{align*}
  s_n &= \alpha s_{n-1} + \gamma b_{n-1} \\
  b_n &= \beta b_{n-1} + \delta s_n
\end{align*}
\]

where \( \alpha, \beta, \gamma, \) and \( \delta \) are given constants.

**Proposition 1.** A necessary condition for the buyer and seller to not come to a stalemate is \( \gamma = 1 - \alpha \) and \( \delta = 1 - \beta \).

**Proof.** Suppose that \( \lim s_n = \lim b_n = P \). Taking the limit of both equations in (3.1) yields \( P = \alpha P + \gamma P \) and \( P = \beta P + \delta P \) which provide the condition after division by \( P \neq 0 \).

\[
s_n - s_{n-1} = (\alpha - 1) s_{n-1} + (1 - \alpha) b_{n-1} = (1 - \alpha) (b_{n-1} - s_{n-1})
\]

\( \blacksquare \)

**Proposition 2.** The coefficients \( \alpha \) and \( \beta \) are both contained in the interval \([0, 1]\).

**Proof.** Since the buyer’s counteroffers are monotonically increasing and the seller’s counteroffers are monotonically decreasing,

\[
b_n - b_{n-1} \geq 0
\]

and

\[
s_n - s_{n-1} \leq 0.
\]

Therefore,

\[
(1 - \beta) (s_n - b_{n-1}) \geq 0
\]

and

\[
(1 - \alpha) (b_{n-1} - s_{n-1}) \leq 0.
\]

For proposed counteroffers, \( s_n \geq b_{n-1} \) and \( b_{n-1} \leq s_{n-1} \). Hence,

\[
\beta \leq 1
\]

and

\[
\alpha \leq 1.
\]

In a similar fashion, it can be shown that \( \beta \geq 0 \) and \( \alpha \geq 0 \). \( \blacksquare \)
It is interesting to see if this model could result in a stalemate. We consider the nontrivial case where \( \alpha, \beta \in (0, 1) \). The seller’s and buyer’s next offers are now at a fixed proportion between the seller’s and buyer’s previous offers given in (3.2) with \( b_0 = 0 \) and \( s_0 = 1 \),

\[
\begin{align*}
  s_n &= \alpha s_{n-1} + (1 - \alpha) b_{n-1} \\
  b_n &= \beta b_{n-1} + (1 - \beta) s_n
\end{align*}
\]  

(3.2)

where \( \alpha \) and \( \beta \) are given constants in \((0, 1)\).

Solving (3.2) yields

\[
b_n = \frac{\alpha (1 - \beta)}{1 - \alpha \beta} - \frac{1 - \beta}{1 - \alpha \beta} \alpha^{n+1} \beta^n
\]

(3.3)

and

\[
s_n = \frac{\alpha (1 - \beta)}{1 - \alpha \beta} + \frac{(1 - \alpha)}{1 - \alpha \beta} \alpha^{n+1} \beta^{n+1}.
\]

(3.4)

Since \( \alpha \beta \in (0, 1) \), we have

\[
\lim s_n = \lim b_n = \frac{\alpha (1 - \beta)}{1 - \alpha \beta}
\]

(3.5)

as the final price that both bargainers tend towards. Therefore, there is no stalemate in this model.

As an aside, this simple model provides an interesting look at how bargainers can tend toward a fair bargain, i.e. one in which \( \lim s_n = \lim b_n = \frac{1}{2} \). Experimental evidence, e.g. Güth et al. [17], Janssen [20], Andreoni and Bernheim [2], and Levati et al. [24], suggests that the 50-50 split is a naturally appealing and preferred bargain for some individuals. Manipulation of (3.5) provides the condition for the 50-50 split as

\[
\frac{1}{\alpha} + \beta = 2.
\]

(3.6)

In particular, if \( \alpha + \beta = 1 \), we have \( \alpha = \frac{1}{\phi} \) and \( \beta = 1 - \frac{1}{\phi} \) where \( \phi \) is the golden ratio, a constant often associated with beauty in nature. Furthermore, myopic buyers and sellers might attempt to create a fair bargain by choosing \( \alpha = \beta = 0.5 \). To the myopic seller’s dismay, inserting these constants into (3.5) yields that the item is sold at \( \frac{1}{2} \) so that the final price is twice as close to the buyer’s initial price. This advantage is distinctly a second-mover advantage that the seller can exploit by forcing the buyer to move first. This is only fair if the buyer’s cost to bargain is higher than the seller’s.

4. Model

The sequential form of the offers is made more general with the system of difference equations

\[
\begin{align*}
  s_n &= \alpha_n s_{n-1} + (1 - \alpha_n) b_{n-1} \\
  b_n &= \beta_n b_{n-1} + (1 - \beta_n) s_n
\end{align*}
\]

(4.1)

where \( \alpha_n \in (0, 1) \) and \( \beta_n \in (0, 1) \). If \( \alpha_n \to 1 \) then the seller is increasingly committed. If \( \beta_n \to 1 \) then the buyer is increasingly committed.

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2 There are several special cases that are straightforward to consider: (i) \( \alpha = 1 \) and \( \beta \in [0, 1) \); (ii) \( \beta = 1 \) and \( \alpha \in [0, 1) \); (iii) \( \beta = 1 \) and \( \alpha = 1 \); and (iv) \( \alpha = 0 \) and \( \beta = 0 \). In (i) the item is sold at a price of 1. In (ii) and (iv) the item is sold at a price of 0. In (iii) there is a stalemate where the buyer only offers 0 and the seller only offers 1. This case corresponds to perfectly committed buyers and sellers.

3 The proof for (3.3) and (3.4) is excluded as this result is a special case of Theorem 1 in Section 3.
**Theorem 1.** The solution to (3.1) for \( n \geq 4 \) is given by

\[
b_n = b_2 + \alpha_1\alpha_2\beta_1 \sum_{k=3}^{n} (1 - \beta_k) \prod_{j=3}^{k} \alpha_j\beta_{j-1}
\]  

(4.2)

and

\[
s_n = b_2 + \alpha_1\alpha_2\beta_1 \left( \prod_{k=3}^{n} \alpha_k\beta_{k-1} + \sum_{k=3}^{n-1} (1 - \beta_k) \prod_{j=3}^{k} \alpha_j\beta_{j-1} \right)
\]  

(4.3)

where \( b_2 = \alpha_1 (1 - \beta_1 + \alpha_2\beta_1 (1 - \beta_2)) \).

**Proof.** From (4.1), we obtain

\[
s_n = \frac{1}{1 - \beta_n} b_n - \frac{\beta_n}{1 - \beta_n} b_{n-1}.
\]  

(4.4)

Using this result in (4.1) yields

\[
\frac{1}{1 - \beta_n} b_n - \left( \frac{\beta_n}{1 - \beta_n} + \frac{\alpha_n}{1 - \beta_{n-1}} + 1 - \alpha_n \right) b_{n-1} + \frac{\alpha_n\beta_{n-1}}{1 - \beta_{n-1}} b_{n-2} = 0.
\]  

(4.5)

After simplification, we have

\[
(1 - \beta_{n-1}) b_n - (1 - \beta_{n-1}) b_{n-1} - \alpha_n\beta_{n-1} (1 - \beta_n) b_{n-1} + \alpha_n\beta_{n-1} (1 - \beta_n) b_{n-2} = 0.
\]  

(4.6)

Re-arranging terms produces

\[
\frac{\Delta b_n}{\Delta b_{n-1}} = \frac{\alpha_n\beta_{n-1} (1 - \beta_n)}{1 - \beta_{n-1}}
\]  

(4.7)

where \( \Delta b_n = b_n - b_{n-1} \) for \( n \geq 2 \).

For \( k \geq 3 \), we now have

\[
\Delta b_k = \Delta b_2 \prod_{j=3}^{k} \frac{\Delta b_j}{\Delta b_{j-1}}
\]  

\[= (b_2 - b_1) \prod_{j=3}^{k} \frac{\alpha_j\beta_{j-1} (1 - \beta_j)}{1 - \beta_{j-1}}
\]  

\[= \alpha_1\alpha_2\beta_1 (1 - \beta_k) \prod_{j=3}^{k} \alpha_j\beta_{j-1}.
\]  

(4.8)

Thus,

\[
b_n = b_2 + \sum_{k=3}^{n} \Delta b_k
\]  

\[= b_2 + \alpha_1\alpha_2\beta_1 \sum_{k=3}^{n} (1 - \beta_k) \prod_{j=3}^{k} \alpha_j\beta_{j-1}.
\]  

(4.9)

Then by inserting (4.9) into (4.4),

\[
s_n = b_2 + \alpha_1\alpha_2\beta_1 \left( \prod_{k=3}^{n} \alpha_k\beta_{k-1} + \sum_{k=3}^{n-1} (1 - \beta_k) \prod_{j=3}^{k} \alpha_j\beta_{j-1} \right).
\]  

(4.10)
Corollary 1. The limits \( \lim b_n = b \) and \( \lim s_n = s \) exist with

\[
b = b_2 + \alpha_1 \alpha_2 \beta_1 \sum_{k=3}^{+\infty} (1 - \beta_k) \prod_{j=3}^{k} \alpha_j \beta_{j-1}
\]

(4.11)

and

\[
s = b_2 + \alpha_1 \alpha_2 \beta_1 \left( \prod_{k=3}^{+\infty} \alpha_k \beta_{k-1} + \sum_{k=3}^{+\infty} (1 - \beta_k) \prod_{j=3}^{k} \alpha_j \beta_{j-1} \right).
\]

(4.12)

Furthermore, \( s \geq b \).

Proof. The sequences \( \{b_n\}_{n=0}^{+\infty} \) and \( \{s_n\}_{n=0}^{+\infty} \) are strictly increasing and strictly decreasing, respectively, as

\[
b_n - b_{n-1} = \alpha_1 \alpha_2 \beta_1 (1 - \beta_n) \prod_{j=3}^{n} \alpha_j \beta_{j-1} > 0
\]

(4.13)

and

\[
s_n - s_{n-1} = \alpha_1 \alpha_2 \beta_1 \beta_{n-1} (\alpha_n - 1) \prod_{k=3}^{n-1} \alpha_k \beta_{k-1} < 0.
\]

(4.14)

Hence, they are monotonic. Since \( \{b_n\}_{n=0}^{+\infty} \) and \( \{s_n\}_{n=0}^{+\infty} \) are monotonic and bounded, they are convergent. Meanwhile,

\[
b_n - b_{n-1} = (1 - \beta_n) (s_n - b_{n-1}).
\]

(4.15)

Since \( 1 - \beta_n > 0 \) and \( b_n - b_{n-1} \geq 0 \),

\[
s_n - b_{n-1} \geq 0.
\]

(4.16)

As \( n \to +\infty \), this becomes

\[
s - b \geq 0.
\]

(4.17)

\[\square\]

Corollary 2. The difference between consecutive offers for \( n \geq 4 \) is

\[
s_n - b_{n-1} = \alpha_1 \alpha_2 \beta_1 \prod_{k=3}^{n} \alpha_k \beta_{k-1}.
\]

(4.18)

Proof. This follows directly from Theorem 2.

\[\square\]

5. Stalemates

Corollary 2 gives conditions for stalemates when bargaining with costless delay. The buyer and seller tend to a stalemate if

\[
\lim (s_n - b_{n-1}) = \alpha_1 \alpha_2 \beta_1 \prod_{n=3}^{+\infty} \alpha_n \beta_{n-1} > 0.
\]

(5.1)

As discussed in Section 2, if \( \alpha_n = \alpha \) and \( \beta_n = \beta \) are constants then \( \lim (s_n - b_{n-1}) = 0 \). Hence, as previously shown, a stalemate is impossible in this case. Several necessary and sufficient conditions for the infinite product in (5.1) to converge are:

4By definition, an infinite product is convergent if and only if \( \prod_{n=n_0}^{+\infty} a_n > 0 \).
Theorem 2. A necessary condition for convergence of the product $\prod_{n=n_0}^{+\infty} a_n$ is $\lim_{n \to \infty} a_n = 1$.

Theorem 3. The product $\prod_{n=n_0}^{+\infty} (1 - a_n)$ is convergent if and only if $\sum_{n=n_0}^{+\infty} a_n$ converges.

Theorem 4. The product $\prod_{n=n_0}^{+\infty} (1 - a_n)$ converges if and only if $\sum_{n=n_0}^{+\infty} \log (1 - a_n)$ converges.

Proofs for these theorems are available in Knopp [22].

It is also useful to construct

$$a_n = 1 - \alpha_n \beta_{n-1}$$

and consider the product

$$\prod_{n=3}^{+\infty} (1 - a_n) = \prod_{n=3}^{+\infty} \alpha_n \beta_{n-1}. \quad (5.3)$$

Since $\alpha_n, \beta_n \in (0,1)$, we have that $a_n \in (0,1)$ as well. These theorems provide several necessary and sufficient conditions for a stalemate to occur, listed below. The contrapositive of each of these provide necessary and sufficient conditions for no stalemate to occur.

**Necessary and sufficient conditions for a stalemate**

1. Theorem 2 gives that a necessary condition for a stalemate is $\lim_{n \to \infty} \alpha_n \beta_{n-1} = 1$.

2. Theorem 3 gives that a necessary and sufficient condition for a stalemate is that $\sum_{n=3}^{+\infty} a_n$ converges.

3. Theorem 4 gives that a necessary and sufficient condition for a stalemate is that $\sum_{n=3}^{+\infty} \log (\alpha_n \beta_{n-1})$ converges.

In Section 6, Example 1 demonstrates two bargainers whose counteroffers satisfy the necessary condition for a stalemate but who reach an agreement. Example 2 demonstrates the existence of a stalemate by considering bargainers that commit to their previous counteroffers at a faster rate than in Example 1. In both examples we consider the case where $\delta_s = \delta_b = 1$ and contrast with other cases.

**6. Examples**

**Example 1.** Consider the case where the buyer and seller propose offers with $\alpha_n = 1 - \frac{1}{4n}$, $\beta_n = 1 - \frac{4}{n}$, $c_b = .55$, and $c_s = .45$. Both buyer and seller are increasingly committed. (4.18) yields

$$s_n - b_{n-1} = \frac{21}{64} \prod_{k=3}^{n} \left(1 - \frac{1}{4k}\right) \left(1 - \frac{1}{2(k-1)}\right). \quad (6.1)$$

(6.1) may be simplified as

$$s_n - b_{n-1} = \frac{\Gamma \left( n + \frac{3}{4} \right) \Gamma \left( n - \frac{1}{4} \right)}{\sqrt{\pi} \Gamma \left( \frac{3}{4} \right) n! (n-1)!}. \quad (6.2)$$

5 This condition makes it clear that increasingly committed buyers and sellers satisfy the necessary condition for a stalemate.
where \( \Gamma(z) = \int_0^{+\infty} x^{z-1}e^{-x}dx \). To find \( \lim(s_n - b_{n-1}) \), we use Stirling’s formula: as \( z \to +\infty \), \( \Gamma(z + 1) \sim \sqrt{2\pi z} \left(\frac{z}{e}\right)^z \) with the property \( \Gamma(n + 1) = n! \) to obtain

\[
s_n - b_{n-1} \sim \frac{e^{\frac{3}{4}} (1 - \frac{1}{4n})^{n+\frac{1}{4}} (1 - \frac{3}{4n})^{n-\frac{1}{4}}}{n^{\frac{3}{2}} (1 - \frac{1}{n})^{n-\frac{1}{4}}}.\]

Then,

\[
\lim \frac{e^{\frac{3}{4}} (1 - \frac{1}{4n})^{n+\frac{1}{4}} (1 - \frac{3}{4n})^{n-\frac{1}{4}}}{n^{\frac{3}{2}} (1 - \frac{1}{n})^{n-\frac{1}{4}}} = 0.
\]

Hence, \( s_n - b_{n-1} \to 0 \) and there is no stalemate.

The case with costless delay is depicted in Figure 6.1. If the seller is able to accurately predict the buyer’s counteroffers, the game ends with the seller accepting the buyer’s seventh counteroffer \( (b_7 \approx .5498) \). Otherwise, the game ends on the buyer’s next turn where the buyer will choose to exit and both parties receive their outside option. This outcome suggests that the uninformed seller would be too committed to their previous offer.

This outcome is highly dependent on costless delay. The buyer and seller utility with \( \delta_s = \delta_b = 0.99 \) is depicted in Figures 6.2(a) and 6.2(b), respectively. Assuming that they are able to perfectly predict one another’s counteroffers, the maximum utility for the buyer and the seller will occur after the buyer’s third counteroffer. The seller accepts this counteroffer and the buyer and the seller have respective utilities \( u^b_3 \approx 0.49 \) and \( u^s_3 \approx 0.48 \). This outcome is approximately fair from the perspective of utility.

**Example 2.** We consider the case where the buyer and the seller propose offers with \( \alpha_n = \beta_n = 1 - \frac{1}{2n^2} \). Rather than assigning outside options at the outset, we will observe how different outside options make their counteroffers appropriate. Both buyer and seller are increasingly committed. (4.18) yields

\[
s_n - b_{n-1} = \frac{3}{16} \prod_{k=3}^{n} \left(1 - \frac{1}{2k^2}\right) \left(1 - \frac{1}{2(k-1)^2}\right).
\]

(6.3)
Figure 6.2: (a) The utility of the buyer for each counteroffer and their outside option. Utility for the seller’s counteroffers are black while utility for the buyer’s counteroffers are red. (b) The utility of the seller for each counteroffer and their outside option. Utility for the seller’s counteroffers are black while utility for the buyer’s counteroffers are red.

We use the infinite product formula provided by 4.3.89 in Zucker [36]:

\[
\sin \frac{x}{\sqrt{2}} = x \prod_{k=1}^{\infty} \left( 1 - \frac{x^2}{\pi^2 k^2} \right).
\]

Therefore,

\[
220 \sin \pi \sqrt{2} = \pi \sqrt{2} \prod_{k=1}^{\infty} \left( 1 - \frac{1}{2k^2} \right). \tag{6.4}
\]

Manipulation of (6.3) then yields

\[
s_n - b_{n-1} = \frac{6}{14} \prod_{k=1}^{n} \left( 1 - \frac{1}{2k^2} \right) \prod_{k=1}^{n+1} \left( 1 - \frac{1}{2k^2} \right). \tag{6.5}
\]

Taking the limit of (6.5) and incorporating (6.4) gives the existence and magnitude of the stalemate for costless delay

\[
s - b = \lim (s_n - b_{n-1}) = \frac{3}{14} \prod_{k=1}^{\infty} \left( 1 - \frac{1}{2k^2} \right) \prod_{k=1}^{\infty} \left( 1 - \frac{1}{2k^2} \right) = \frac{12}{7} \pi \sin^2 \frac{\pi}{\sqrt{2}}. \tag{6.6}
\]

Figure 6.3 depicts the first 20 buyer and seller counteroffers with costless delay. We observe that if \( c_b \approx 0.47 \) and \( c_s \approx 0.7 \) then these sequences of counteroffers are appropriate as the counteroffers tend towards the outside option. This exemplifies the prior insight that buyers and sellers grow increasingly committed to prior offers when they have relatively strong outside options. In this case, the outside options are so strong that they are disparate with one another and these bargainers are not appropriate for bargaining with one another.

Figures 6.4(a) and 6.4(b) depict the buyer’s and seller’s respective utilities from the first 10 counteroffers with \( \delta_s = \delta_b = 0.98 \). We observe that if \( c_b = 0.55 \) and \( c_s = 0.4 \) then the buyer and seller can come to an agreement. This shows that the impact of stalemates might diminish with costly delay. The buyer will now accept the seller’s third counteroffer and the buyer and the seller receive respective utilities \( u^b_3 \approx 0.52 \) and \( u^s_3 \approx 0.43 \). In this case, the outside options and delay are such that the bargainers receive higher utility from bargaining. This is in spite of the fact that they are increasingly committed.

7. Conclusion

This paper develops game theoretic fundamentals for a bilateral bargain in which players make decisions based on limited lookahead rather than backward induction. Each player may propose a counteroffer, accept their opponent’s previous
counteroffer, or exit and receive an outside option. They make the decision that maximizes expected utility for their next turn. Each player learns in the game through a sequence of conditional probability distributions regarding the other’s future decisions. Certain basic facts regarding bargainer behavior are gleaned from this framework.

The bargaining game motivates the use of systems of difference equations to model the bargainers’ counteroffers. The use of difference equations to describe counteroffers provides a tractable framework that can be extended to analyze a multitude of bargainer types. A preliminary model is developed in which bargainers propose counteroffers at a fixed proportion between previous counteroffers. The preliminary model demonstrates basic properties of bargaining. Stalemates never occur in the preliminary model except in a special case. This model is generalized to demonstrate that more dynamic bargainers can tend to an impasse. Conditions for a stalemate to occur are given. The conditions reveal that stalemates occur when bargainers grow increasingly committed to their prior offers. Several examples demonstrate that bargaining is at times mutually beneficial. With costless delay, stalemates signal that each bargainer’s outside option is too strong for the other to benefit. When delay is costly, the impact of stalemates might diminish for particular outside options.


