The Method of Fundamental Solutions for Solving the Axisymmetric Poisson Equation

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Summary

A mesh free numerical scheme using the method of fundamental solutions (MFS) has been developed to solve the axisymmetric Poisson problem. The method of particular solutions has been employed to split the solution into a particular solution and homogeneous solution. To evaluate the particular solution, Chebyshev polynomial are employed to interpolate the source term of the Poisson equation and a closed form particular solution for monomial terms is derived. We then use the MFS to obtain the solution of the homogeneous equation.

Introduction

Solving partial differential equations using boundary methods such as boundary element methods, Trefftz methods, and the method of fundamental solution is of considerable interest in the science and engineering communities. The major obstacle to doing this is the necessity for computing particular solutions of the inhomogeneous PDE. Although, there are known analytical formulas for particular solutions in the form of generalized Newtonian potentials, these are typically difficult to evaluate numerically. In the engineering literature, starting with the work of Nardini and Brebbia [5], the most popular method for overcoming these difficulties has been to approximate the source term by a suitable class of basis functions and then find a particular solution for each basis function. For commonly occurring operators such as the Laplacian and Helmholtz-type operators, the usual choice has been radial basis functions (\textit{rbfs}). However, it can be difficult to find highly accurate \textit{rbf} approximations to the source term, and the resulting numerical computation can be very ill-conditioned. To mitigate some of these difficulties, the authors have recently examined the use of polynomial approximations [3]. Since these approximations can achieve spectral accuracy and can be obtained without matrix inversion, some of the problems that occur when using \textit{rbfs} can be largely overcome. However, as shown in [3], these calculations can be extremely complex and time consuming in 3-D. It is the goal of this paper to consider possible simplifications from considering problems with axisymmetric geometry.

In [7], it was necessary to solve Poisson’s equation with axisymmetric geometry and axisymmetric data. As is well known, for Laplace’s equation, this gives a substantial dimensionality reduction providing the boundary conditions are axisymmetric, as well. Since the axisymmetric Laplacian does not have constant coefficients, the standard technique of radial basis functions approximation of source terms is not easily implemented. In [7], Wang attempts such an approximation by using a 3-D \textit{rbf} interpolation and then finding particular solutions by integrating over the azimuthal angle. However, this tends to offset the effects of the axisymmetry and analytical formulas for the particular solutions are difficult to obtain. Although one can avoid the need for analytical formulas by numerical integration, this further decreases the efficiency of the algorithm.

In this paper, we are able to find analytical formulas by using 2-D polynomial approximations. In addition, we are able to further improve the accuracy by using the MFS rather than the standard BEM to solve the resulting Laplace equation.

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The method of particular solutions

For simplicity, we consider the numerical solution of the Poisson problem

\[ \Delta u(P) = f(P), \quad P \in D \tag{1} \]
\[ u(P) = g(P), \quad P \in S, \tag{2} \]

where \( \Delta \) denotes the Laplacian and \( D \) is a bounded domain in \( \mathbb{R}^3 \) with boundary \( S \), that we assume to be piecewise smooth. The function \( f(P) \) is a smooth source term. Let the region \( D \) be axisymmetric. Then, (1) reduces to

\[ \frac{\partial^2 u(P)}{\partial r^2} + \frac{1}{r} \frac{\partial u(P)}{\partial r} + \frac{\partial^2 u(P)}{\partial z^2} = f(P), \quad P \in \Omega. \tag{3} \]

When the boundary conditions and \( f(P) \) are also axisymmetric, the three-dimensional problem reduces to solving the axisymmetric version of Poisson’s equation (3) with boundary conditions (2).

To find the solution \( u \), we take the usual steps:

(i) First find a particular solution \( u_p : \Delta u_p = f, \) in \( S \).
(ii) Then solve the homogeneous problem

\[ \Delta v = 0 \quad \text{in} \ D, \quad v = g - u_p \quad \text{on} \ S. \tag{4} \]

(iii) Get the answer to the problem \( u = v + u_p \).

We consider approximating \( f \) in (3) by a polynomial in \( r \) and \( z \) and then obtaining particular solutions by analytically finding a particular solution to

\[ \frac{\partial^2 u(P)}{\partial r^2} + \frac{1}{r} \frac{\partial u(P)}{\partial r} + \frac{\partial^2 u(P)}{\partial z^2} = r^j z^k, \quad j, k \geq 0 \tag{5} \]

This approach was briefly considered and discussed by Wang in [7]. Without going into the details, the principle obstacle there was the inability to obtain sufficiently accurate polynomial approximations to \( f \).

Polynomial approximation

Because one generally cannot find polynomial interpolants in \( \mathbb{R}^d, \ d \geq 2 \), by interpolating at arbitrary sets of points as one can do for rbfs, one must use a somewhat different approach. Here, we briefly review the method used in [3] for the non-axisymmetric case.

We first embed the domain \( \Omega \) in a rectangle \( \hat{\Omega} \) in the \( r - z \) plane. Assume that \( \hat{\Omega} \) is the domain \([0,a] \times [b,c]\). Then \( f \) is approximated by a sum of products of Chebyshev polynomials in \( r \) and \( z \) by interpolating \( f \) on the pseudo-spectral points in \( \hat{\Omega} \). Then, using a symbolic program such as MATHEMATICA or MAPLE, this polynomial is reexpanded in monomial form. This gives an approximation of the form

\[ \hat{f}(r,z) = \sum_{j=0}^m \sum_{k=0}^n a_{jk} r^j z^k. \tag{6} \]
Then an approximate particular solution is found as

\[ u_p(r,z) = \sum_{k=0}^{m} \sum_{j=0}^{n} a_{jk} \psi_{jk}(r,z) \]  

(7)

where \( \psi_{jk} \) is a solution to (5).

Chebyshev interpolation in 2D

Multi-dimensional polynomial interpolants are difficult to obtain and may not exist for arbitrary finite subsets of points in \( \mathbb{R}^d \), \( d \geq 2 \). In fact, that difficulty has occurred in previous work when using polynomial particular solutions in the DRM [1]. However, if the points are chosen on a rectangular grid in \( \mathbb{R}^2 \) and a box grid in \( \mathbb{R}^3 \), then one can find Lagrange interpolants by using the forms of the one dimensional interpolants [3].

In \( \mathbb{R}^2 \), we consider interpolating \( f(x,y) \) on the set \( \{(x_j, y_k)\}, 0 \leq j \leq m, 0 \leq k \leq n \). It is straightforward to verify that

\[ q_{m,n}(x,y) = \sum_{k=0}^{n} \sum_{j=0}^{m} f(x_j, y_k) l_j(x) l_k(y) \]  

(8)

interpolates to \( f(x,y) \) at \( \{(x_j, y_k)\}, 0 \leq j \leq m, 0 \leq k \leq n \). Here, \( l_j(x) \) and \( l_k(y) \) are the fundamental Lagrange polynomial interpolants on \( \{x_j\}_{j=0}^{m} \) and \( \{y_k\}_{k=0}^{n} \) respectively. That is,

\[ q_{m,n}(x_j, y_k) = f(x_j, y_k), \quad 0 \leq j \leq m, \quad 0 \leq k \leq n. \]  

(9)

As in the one dimensional case, it is convenient to choose \( \{x_j\}_{j=0}^{m} \) and \( \{y_k\}_{k=0}^{n} \) as the images of the pseudo-spectral points in \([a, b] \times [c, d] \). Then,

\[ q_{m,n}(x,y) = \sum_{k=0}^{n} \sum_{j=0}^{m} a_{jk} T_j \left( \frac{2x - b - a}{b - a} \right) T_k \left( \frac{2y - d - c}{d - c} \right) \]  

(10)

where

\[ a_{jk} = \frac{4}{nm\bar{c}^n} \sum_{q=0}^{n} \sum_{p=0}^{m} \frac{f(x_p, y_q)}{c_p c_q} \cos \left( \frac{\pi pq}{n} \right) \cos \left( \frac{\pi qk}{m} \right). \]  

(11)

In the remaining sections, we will use \( (r,z) \) instead of \( (x,y) \), the conventional notation, (8)-(11), where \( r, \theta, \) and \( z \) represent cylindrical coordinates.

Derivation of particular solutions

Once a polynomial approximation \( \hat{f} \) to \( f \) is obtained, an approximate particular solution to the axisymmetric Poisson equation

\[ \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = \hat{f} \]  

(12)
can be obtained by solving (3) and then summing the results as in (6). The result is given in Theorem 1 below.

**Theorem 1.** Let \(k\) and \(m\) be integers, \(k \geq 0, m \geq 0\), and denote by \([x]\) the largest integer less than or equal to \(x\). Then a particular solution to

\[
\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = r^k z^m
\]

(13)

is given by

\[
u = \sum_{j=0}^{[m/2]} \left(-1\right)^j m! \left[ \frac{k!!}{(k+2j+2)!!} \right]^2 z^{m-2j} r^{k+2j+2} \]

(14)

where \(0!! = 1, 1!! = 1, 2!! = 2\), and

\[
k!! = \begin{cases} 
2 \cdot 4 \cdot 6 \cdots k, & \text{if } k \text{ is an even positive integer } (k > 2), \\
1 \cdot 3 \cdot 5 \cdots k, & \text{if } k \text{ is an odd positive integer } (k > 1). 
\end{cases}
\]

(15)

**Proof:** Anticipating (14), we look for a particular solution of (13) of the form

\[
u = \sum_{j=0}^{[m/2]} A_j z^{m-2j} r^{k+2j+2} \]

(16)

Then we calculate \(\frac{\partial u}{\partial r}, \frac{\partial^2 u}{\partial r^2}, \) and \(\frac{\partial^2 u}{\partial z^2}\) and plug them into (13). We obtain

\[
(k+2)^2 A_0 z^k + \sum_{j=0}^{[m/2]} (k+2j+2)^2 A_j z^{m-2j} r^{k+2j} \\
+ \sum_{j=0}^{[m/2]} (m-2j+2)(m-2j+1) A_{j-1} z^{m-2j} r^{k+2j} - z^m r^k.
\]

(17)

Now, equating the like terms on both sides of (17) gives

\[
A_0 = \frac{1}{(k+2)^2}
\]

(18)

\[
(k+2j+2)^2 A_j + (m-2j+2)(m-2j+1) A_{j-1} = 0, \quad j = 1, 2, \cdots, \left[\frac{m}{2}\right].
\]

(19)

Hence, finding \(\frac{A_j}{A_{j-1}}\) from (19), we get

\[
A_j = A_0 \prod_{\ell=1}^{j} \frac{A_{\ell}}{A_{\ell-1}} = \frac{1}{(k+2)^2} \prod_{\ell=1}^{j} \frac{(-1)(m-2\ell+2)(m-2\ell+1)}{(k+\ell+2)^2}
\]

(20)
Some manipulation of (20) gives

\[
A_j = \frac{(-1)^j m![(k + 2)!]!^2}{(k + 2)^2 (m - 2j)![(k + 2j + 2)!]!^2} = \frac{(-1)^j m![(k!!)!^2}{(m - 2j)![(k + 2j + 2)!]!^2}
\]

(21)

Substituting (21) into (16) concludes the proof of Theorem 1.

The axisymmetric MFS

As indicated in Section 2, once we have obtained a particular solution to the Poisson’s equation (3), an approximation to the BVP (1)-(2) can be obtained by solving (4) and then

\[
u = \hat{v} + u_p.
\]

(22)

In [7], Wang used a standard boundary element method to obtain v. In this paper, we follow the approach of Karageorghis and Fairweather to solve (4) by using an axisymmetric version of the MFS.

In the usual formulation of the MFS to solve BVPs for the 3D Laplacian, \(m\) points \(\{P_j\}_{j=1}^M\) are chosen in an extension \(\hat{S}\) of \(S\) and \(v\) is approximated by

\[
v(Q) \simeq \hat{v}(Q) = \sum_{\ell=1}^m a_k G(P_k, Q),
\]

(23)

where \(G(P, Q) = 1/\|P - Q\|\) is the fundamental solution for the Laplacian. To satisfy the boundary condition (4), \(m\) points \(\{Q_j\}_{j=1}^N\) are chosen on \(S\) and then setting

\[
\hat{v}(Q_k) = v(Q_k), \quad k = 1, 2, ..., m.
\]

(24)

(23) gives

\[
\sum_{j=1}^m a_j G(P_j, Q_k) = v(P_k), \quad k = 1, 2, ..., m.
\]

(25)

Solving (25) for \(\{a_j\}_{j=1}^m\) gives an approximation \(\hat{v}\) to \(v\).

If \(D\) is axisymmetric, then to determine an axisymmetric solution, (25) can be modified in the following way. Let \(P = (r_P, 0, z_P)\) be a point in \(\Omega\), the generating domain for \(D\) and let \(Q = (r_Q \cos \theta, r_Q \sin \theta, z_Q)\). Then integrating \(G(P, Q)\) over \(\theta\), we get the axisymmetric fundamental solution

\[
G(A(P, Q)) = \frac{4K(\kappa)}{R}
\]

(26)

where again \(K(\kappa)\) is the complete elliptic integral of first kind,

\[
R = \left[ (r_P + r_Q)^2 + (z_P - z_Q)^2 \right]^{1/2}, \quad \kappa^2 = \frac{4r_pr_Q}{R^2}.
\]

(27)
Also, the normal derivative \( \frac{\partial G(P,Q)}{\partial n_Q} \) is given by [4]

\[
\frac{\partial G(P,Q)}{\partial n_Q} = -\frac{4(z_Q - z_P)E(\kappa)}{R^3(1 - \kappa^2)} n_z \\
+ 2 \left\{ \frac{R^2}{r_Q R^3(1 - \kappa^2)} \left[ E(\kappa) - K(\kappa) \left( 1 - \kappa^2 \right) \right] - 2r_2(r_Q + r_P)E(\kappa) \right\} n_r
\]

where \( E(\kappa) \) is the complete elliptic integral of the second kind and \( n_r, n_z \) are the components of the outward normal to \( S \) in the \( r \) and \( z \) directions.

To obtain the axisymmetric MFS, \( m \) points \( \{P_j\}_{j=1}^m \) are chosen outside of the domain \( \Omega \) and then \( v \) is approximated by \( \hat{v} \) where

\[
\hat{v}(Q) = \sum_{j=1}^{m} a_j G_A(P_j, Q), \quad Q \in \bar{\Omega},
\]

where \( \bar{\Omega} \) is the closure of \( \Omega \). To determine \( \{a_j\}_{j=1}^m \), \( m \) points \( \{Q_k\}_{k=1}^m \) are chosen on the physical boundary and then we set

\[
\hat{v}(Q_k) = v(Q_k), \quad k = 1, 2, \ldots, m.
\]

This gives

\[
\sum_{j=1}^{m} a_j G_A(P_j, Q_k) = v(Q_k), \quad k = 1, 2, \ldots, m.
\]

We note that our method for choosing \( \{Q_k\} \) differs from that used in [4].

**Numerical results**

To demonstrate the effectiveness of our numerical algorithm, we consider one axisymmetric problem. We choose the symbolic software MATHEMATICA for coding. All the problems in this section were run on a Notebook PC with a 650MHZ Pentium III processor.

**Example 1.** Consider the axisymmetric Poisson’s Problem from Wang’s thesis [7],

\[
\frac{\partial^2 u(r,z)}{\partial r^2} + \frac{1}{r} \frac{\partial u(r,z)}{\partial r} + \frac{\partial^2 u(r,z)}{\partial z^2} = -\frac{\sin r}{r e^z}, \quad P \in D,
\]

\[
u(r,0) = \cos r, \quad u(r,1) = e^{-1} \cos r, \quad u(1,z) = e^{-z} \cos 1.
\]

where \( D = \{(r,z) : 0 < r < 1, 0 < z < 1\} \). The exact solution of the above axisymmetric problem is given by \( u = e^{-z} \cos r \). For solving the homogenous equation using the MFS, we choose 17 collocation points on the boundary and the same number of source points evenly distributed on three sides of the
To evaluate the particular solution, we choose various values of Gauss-Lobatto nodes. We denote $M$ as the number of Gauss-Lobatto points in the $r$ and $z$ directions. The numerical tests were performed on 100 evenly distributed points in the domain or on its boundary. The absolute maximum error in Table 1 demonstrates the robustness of our proposed numerical scheme.

<table>
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<th>$L_{\infty}$</th>
<th>$M$</th>
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<tr>
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Conclusions

In this paper, we have shown how to extend the MFS for solving the axisymmetric Laplace equation in $\mathbb{R}^3$ to the axisymmetric Poisson equation. To do this, the source term is approximated by two dimensional polynomials in cylindrical coordinates. A Chebyshev interpolation scheme is employed to obtain the approximations which are spectrally convergent for smooth data. The resulting numerical algorithm is shown to be rapidly convergent by applying it to a number of problems taken from [4] and [7].

In future work, we plan on extending this approach to the axisymmetric Helmholtz equation which arises from discretizing diffusion equations in time. This should then enable us to provide an efficient solver for the axisymmetric Stokes equation [4], [7].

Reference