ON THE SOLUTION OF STEADY UNCONFINED SEEPAGE TOWARDS SEMI-INFINITE SLOPES

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SUMMARY

Analytical solutions for the two-dimensional problem of unconfined seepage towards semi-infinite slopes are described. The analysis employs complex variable techniques. Conformal mappings have been used to reduce the problem to solving an ordinary differential equation. The resulting integrals are presented in terms of special functions. The solutions for the location of the phreatic surface and the exit point have been obtained in parametric form. Some additional results and asymptotic expansions of the solutions are also presented. The numerical results have been calculated and plotted for different values of the slope angle in dimensionless co-ordinate space.

INTRODUCTION

Unconfined seepage towards slopes is an important problem in highway and agricultural engineering and in the engineering of embankments and dams. Since seepage emerging on a slope frequently causes instability of the slope-forming earth mass, adequate analysis of seepage is crucial for proper design of slopes. Analysis of unconfined seepage is commonly handled by approximate methods. However, many of the available approximate solution schemes are inefficient or plagued with difficulties. On the other hand, analytical solutions are particularly attractive because of the general nature of these solutions. Furthermore, exact solutions are useful for calibrating numerical solutions and may be used to incorporate improved features in numerical treatment schemes.

This paper presents the complete development of an analytical solution for steady unconfined seepage towards a semi-infinite slope. The present analysis is planar and the two-dimensional flow region is assumed to be hydrogeologically homogeneous and isotropic. Steady seepage is assumed to occur from infinity at one end towards a semi-infinite slope which intersects a horizontal ground surface at the other end.

The steady-state solution developed in this paper utilizes the technique of conformal transformations. This is a powerful analytical technique for solving two-dimensional problems involving steady confined or unconfined seepage and is well documented in several textbooks. Some previous works which develop partial solutions for related problems, namely free-surface flow towards vertical cuts and slopes, are based on this approach. The purpose of this paper is to present the framework for this kind of analysis to the general problem of steady-state unconfined flow through a homogeneous slope-forming soil mass with any arbitrary slope.
FORMULATION

A two-dimensional problem of unconfined flow in a semi-infinite porous medium as shown in Figure 1 is considered herein. Since the aim is to find an analytical solution, it is convenient to assume the slope-forming mass to be saturated, homogeneous and hydrogeologically isotropic and of infinite lateral extent on either side. In terms of (reduced) complex potentials* (velocity potential function \( \phi(x, y) \) or streamfunction \( \psi(x, y) \)) the problem of steady seepage through the domain (Figure 1) can be formulated as

\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \quad (x, y) \in (-\infty, x_1) \otimes (-\infty, f(x)) \cup [x, 0) \otimes (-\infty, -x \cot(\pi \alpha)]
\]

\[
\cup [0, +\infty) \otimes (-\infty, 0) \uparrow
\]

\[
\phi(x, 0) = 0, \quad 0 \leq x < \infty
\]

\[
\phi(-y \tan(\pi \alpha), y) = y, \quad 0 \leq y \leq -x_1 \cot(\pi \alpha)
\]

\[
\phi(x, f(x)) = f(x), \quad -\infty < x \leq x_1
\]

\[
\frac{\partial \phi}{\partial y} (x, f(x)) = \frac{df(x)}{dx} \frac{\partial \phi}{\partial x} (x, f(x)), \quad -\infty < x \leq x_1
\]

\[
f(x) > 0, \quad -\infty < x \leq x_1
\]

\[
\frac{df(x)}{dx} < 0, \quad -\infty < x \leq x_1
\]

Here \( \phi(x, y) \) and \( f(x) \) are unknown functions, \( x_1 \) is a negative solution of the equation

\[
x = -f(x) \tan(\pi \alpha) \quad (0 < \alpha < \frac{1}{2})
\]

![Figure 1. Flow domain in complex physical (z)-plane](image)

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*Complex potential \( \omega = \phi + i\psi \); \( \phi(x, y) \) and \( \psi(x, y) \) are conjugate harmonic functions.

† \( \otimes \) denotes Cartesian multiplication of sets.
and the reduced potential $\phi(x, y)$ is related to the velocity potential $\phi_s(x, y)$ by

$$\phi = (K - \phi_s)/k$$

(1i)

where $K$ is an arbitrary constant and $k$ is the coefficient of permeability of the medium.

It is worth mentioning that the approach for solving the case of $x = 0$, i.e., a vertical slope, is methodologically different from the general case and has been presented earlier.\(^{11}\)

An equivalent formulation of the problem (1a)-(1g) (Figure 1) is given by

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad \text{inside } 0120$$

(2a)

$$\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \quad \text{inside } 0120$$

(2b)

$$\phi = 0 \quad \text{on } 02$$

(2c)

$$\phi = y \quad \text{on } 01$$

(2d)

$$\phi = y \quad \text{and } \psi = 0 \quad \text{on } 12$$

(2e)

where the functions $\phi(x, y)$ and $\psi(x, y)$ and the shape of the curve 12 are unknown.

The method of conformal mapping is employed to solve this problem. Traditionally, the method involves finding a conformal transformation between the domain in the complex potential ($\omega = \phi + i\psi$) plane (or $\omega$-plane) and the image of the flow domain in the complex plane of an auxiliary function. Figure 2 shows the image of the domain in the $\omega$-plane. Since in the present case the domain is not completely defined $a$ priori in either the $z$- or the $\omega$-plane, the outlook for solving such a problem appears to be hopeless. Fortunately, however, it is possible to make use of two auxiliary functions and mapping on the corresponding planes to solve the problem. The following discussion shows how successive conformal mappings on a number of auxiliary planes are employed. The first two of these functions are Kirchhoff's function $W$ and Zhukovsky's function $U$.

Figure 2. Flow domain in complex potential ($\omega$) plane
Kirchoff's function $W$ is defined as

$$W = \frac{dz}{d\phi} = \frac{dx + idy}{d\phi + id\psi} = \Phi + i\Psi$$

(3)

The flow domain can be represented on the $W$-plane as shown in Figure 3. The conditions on the domain boundaries can be written as follows.

On 02:

$$W = -i \frac{dx}{d\psi} \text{ since } \phi = 0 \text{ and } y = 0$$

i.e. $\Phi = 0$ and $\Psi < 0$

(4a)

(4b)

On 12:

$$W = \frac{dx}{dy} + i \text{ since } \phi = y \text{ and } \psi > 0$$

i.e. $\Phi > 0$ and $\Psi = 0$

(4c)

(4d)

On 01:

$$W = \frac{idy - dy \tan(\pi \alpha)}{dy + id\psi} \text{ since } \phi = y \text{ and } x = -y\tan(\pi \alpha)$$

(4e)

$$W = \frac{i - \tan(\pi \alpha)}{1 + it} \text{ if } t = \frac{d\psi}{dy}$$

(4f)

i.e. $\Phi = \frac{t - \tan(\pi \alpha)}{t^2 + 1}$ and $\Psi = \frac{1 + t\tan(\pi \alpha)}{t^2 + 1}$

(4g)

Figure 3. Transformation of flow domain on $W$-plane
so that

$$[\Phi + \frac{1}{2} \tan(\pi x)]^2 + (\Psi - \frac{1}{2})^2 = \left(\frac{1}{2\cos(\pi x)}\right)^2 \quad (4h)$$

Zhukovsky's function $U$, on the other hand, is defined as

$$U = z - i\omega = (x + \psi) + i(y - \phi) = \Phi^* + i\Psi^* \quad (5)$$

Substituting the given conditions on the boundaries in a similar manner, the flow domain can be represented as a half-plane in the $U$-plane as shown in Figure 4.

**ANALYTICAL SOLUTION**

A complex function $\zeta$ may be defined as

$$\zeta = -\frac{1}{W-i}\tan(\pi x) \quad (6)$$

This function $\zeta$ can also be written as

$$\zeta = -i\frac{d\omega}{dU}\tan(\pi x) \quad (7)$$

Clearly, from equation (6), the domain shown in Figure 3 maps onto the interior of a triangle in the $\zeta$-plane. The image of the flow domain in the $\zeta$-plane is shown in Figure 5. Now another complex function $\kappa$ is defined as

$$\kappa = \psi_0 - U \quad (8)$$

The triangular domain in the $\zeta$-plane can be mapped onto the upper $\kappa$-half-plane (as shown in Figure 6) using the Schwarz–Christoffel integral which is given by

$$\zeta = M \int_0^\kappa \kappa^{-1/2} - (\kappa_1 - \tilde{\kappa})^{1/2} d\tilde{\kappa} + N \quad (9)$$

Figure 4. Transformation of flow domain on $U$-plane
where

\[ \kappa_1 = \psi_0 + r \sin(\pi \alpha) \]

From the prescribed conditions at points 1 and 0, the constants \( M \) and \( N \) can be evaluated as

\[ M = \frac{[i - \tan(\pi \alpha)] \sqrt{\pi \kappa_1}}{\Gamma(\alpha) \Gamma(\frac{1}{2} - \alpha)} \quad (10a) \]

\[ N = \tan(\pi \alpha) \quad (10b) \]

Using equations (7), (9), (10a) and (10b), it can be shown that

\[ -i \frac{d \omega}{d U} \tan(\pi \alpha) = \tan(\pi \alpha) + [i - \tan(\pi \alpha)] \frac{i \sqrt{\pi \kappa_1}}{\Gamma(\alpha) \Gamma(\frac{1}{2} - \alpha)} \int_{0}^{\phi_0 - U} \kappa^{-1/2 - \varepsilon} (\kappa_1 - \kappa)^{\varepsilon - 1} d\kappa \quad (11a) \]

Hence one obtains

\[ \frac{dz}{dU} = -i \frac{\sqrt{\pi \kappa_1}}{\Gamma(\alpha) \Gamma(\frac{1}{2} - \alpha) \sin(\pi \alpha)} \int_{0}^{\phi_0 - U} \kappa^{-1/2 - \varepsilon} (\kappa_1 - \kappa)^{\varepsilon - 1} d\kappa \quad (11b) \]
Now, by letting $\kappa = \kappa_1 \sigma$, integration of equation (11b) yields

$$z = \frac{i \sqrt{\pi} e^{i \alpha \pi}}{\Gamma(\alpha) \Gamma\left(\frac{1}{2} - \alpha\right) \sin(\pi \alpha)} \int_0^{\phi_0} \int_0^\tau \sigma^{-1/2 - \alpha}(1 - \sigma)^{\alpha - 1} d \sigma$$

where

$$\tau = \frac{\psi_0 - \xi}{\psi_0 + r \sin(\pi \alpha)}$$

Since at point 1 (Figures 1 and 4), $z = i r e^{i \alpha \pi}$ and $U = -r \sin(\pi \alpha)$, one obtains

$$i r e^{i \alpha \pi} = \frac{i \sqrt{\pi} e^{i \alpha \pi}}{\Gamma(\alpha) \Gamma\left(\frac{1}{2} - \alpha\right) \sin(\pi \alpha)} \int_0^{\phi_0} \int_0^\tau \sigma^{-1/2 - \alpha}(1 - \sigma)^{\alpha - 1} d \sigma$$

Equation (13) can be recast in the form

$$r \sin(\pi \alpha) = \frac{\sqrt{\pi} \left[ \psi_0 + r \sin(\pi \alpha) \right]}{\Gamma(\alpha) \Gamma\left(\frac{1}{2} - \alpha\right)} \int_0^1 \int_0^\tau \sigma^{-1/2 - \alpha}(1 - \sigma)^{\alpha - 1} d \sigma$$

The integral in equation (14a) is evaluated in Appendix II. Equation (73) allows one to write equation (14a) as

$$r \sin(\pi \alpha) = \frac{\sqrt{\pi} \left[ \psi_0 + r \sin(\pi \alpha) \right]}{\Gamma(\alpha) \Gamma\left(\frac{1}{2} - \alpha\right)} \frac{2 \alpha}{\sqrt{\pi} \Gamma(\alpha) \Gamma\left(\frac{1}{2} - \alpha\right)}$$

Hence the parameter $\psi_0$ can be obtained as

$$\frac{\psi_0}{r} = \frac{1 - 2 \alpha}{2 \alpha}$$

which gives

$$\frac{\psi_0}{\psi_0 - \frac{2 \alpha - 1}{2 \alpha}}$$

Equation (12) can therefore be written as

$$\frac{2z}{\psi_0} = \frac{i \sqrt{\pi} e^{i \alpha \pi}}{\Gamma(\alpha) \Gamma\left(\frac{1}{2} - \alpha\right) \sin(\pi \alpha)} \int_0^u \int_0^\tau \sigma^{-1/2 - \alpha}(1 - \sigma)^{\alpha - 1} d \sigma$$

where

$$u = (1 - 2 \alpha) \left( 1 - \frac{U}{\psi_0} \right)$$

Equation (17a) represents the solution of the problem at hand in an integral form and is the same as that presented by Aravin and Numerov. The integral on the right-hand side of equation (17a) is evaluated in Appendix II. Using the results obtained in equations (73), (78) and (80), equation (17a) reduces to

$$\frac{2z}{\psi_0} = \frac{2i e^{i \alpha \pi} U}{\sin(\pi \alpha) \psi_0} \frac{i \sqrt{\pi} I_5}{\Gamma(\alpha) \Gamma\left(\frac{1}{2} - \alpha\right) \sin(\pi \alpha)}$$

where

$$I_5 = \int_1^u \int_1^\tau \sigma^{-1/2 - \alpha}(\sigma - 1)^{\alpha - 1} d \sigma$$
Equation (18a) then simplifies to
\[ z = U [1 - \cot(\pi \alpha)] - \frac{i \psi_0 \sqrt{(\pi)} I_5}{2 \Gamma(\alpha) \Gamma(\frac{3}{2} - \alpha) \sin(\pi \alpha)} \] (19)

Since \( \Gamma(\alpha) \Gamma(1 - \alpha) = \pi / \sin(\pi \alpha) \), equation (19) can be further simplified to
\[ \omega = -U \cot(\pi \alpha) - \frac{i \psi_0}{2 \sqrt{(\pi)}} \frac{\Gamma(1 - \alpha)}{\Gamma(\frac{3}{2} - \alpha)} I_5 \] (20)

Finally, substituting the value of the integral \( I_5 \), from equation (88) and rearranging terms, the solution to the problem can be expressed as
\[ \frac{\omega}{\psi_0} = \frac{1}{\sqrt{(\pi \alpha)}} \frac{\Gamma(1 - \alpha)}{\Gamma(\frac{3}{2} - \alpha)} \left[ \left(1 - \frac{1}{u}\right)^{\frac{3}{2}} + (u + 2\alpha - 1) F\left(\frac{1}{2}; 1 - \alpha; \frac{3}{2}; 1/u\right) \right] \] (21a)

where
\[ u = (1 - 2\alpha) \left(1 + \frac{i \omega - z}{\psi_0}\right) \] (21b)

**ADDITIONAL RESULTS**

**Equation of phreatic curve**

Since \( \phi = y \) and \( \psi = 0 \) on the phreatic surface, the values of the complex potential \( \omega \) and Zhukovsky's function \( U \) on the phreatic surface are given by
\[ \omega = y, \quad U = x \] (22)

Hence the value of the parameter \( u \) on the phreatic surface, \( A \), is given by
\[ A = (1 - 2\alpha) \left(1 - \frac{x}{\psi_0}\right) \] (23)

Therefore, from equation (21), the equation of the phreatic surface can be obtained as
\[ \frac{y}{\psi_0} = \frac{1}{\sqrt{(\pi \alpha)}} \frac{\Gamma(1 - \alpha)}{\Gamma(\frac{3}{2} - \alpha)} \left[ A \left(1 - \frac{1}{A}\right)^{\frac{3}{2}} + (A + 2\alpha - 1) F\left(\frac{1}{2}; 1 - \alpha; \frac{3}{2}; 1/A\right) \right] \] (24)

Alternatively, the equation of the phreatic curve can be expressed in the same form as that given by Aravin and Numerov:11
\[ 2 \sqrt{(\pi)} \frac{y}{\psi_0} \frac{\Gamma(\frac{3}{2} - \alpha)}{\Gamma(1 - \alpha)} = 2 \sqrt{(A)} \left(1 - \frac{1}{A}\right)^{\frac{3}{2}} + (A + 2\alpha - 1) \int_0^{1/A} \rho^{-1/2} (1 - \rho)^{\alpha - 1} d\rho \] (25)

**Slope of tangent to phreatic curve**

Introducing \( B = 1/A \) in equation (25), one obtains
\[ 2 \sqrt{(\pi)} \frac{y}{\psi_0} \frac{\Gamma(\frac{3}{2} - \alpha)}{\Gamma(1 - \alpha)} = 2 \left(1 - B^x\right) \left(\frac{1}{B} + 2\alpha - 1\right) \int_0^B \rho^{-1/2} (1 - \rho)^{\alpha - 1} d\rho \] (26)

Differentiation of the above equation with respect to \( x \) gives
\[ \frac{dy}{dx} = \frac{\psi_0}{2 \sqrt{(\pi)}} \frac{\Gamma(1 - \alpha)}{\Gamma(\frac{3}{2} - \alpha)} \frac{dA}{dx} \int_0^{1/A} \rho^{-1/2} (1 - \rho)^{\alpha - 1} d\rho \] (27)
Since $\frac{dA}{dx} = -(2/\psi_0)(\frac{1}{4} - \alpha)$, equation (27) becomes
\[
\frac{dy}{dx} = -\frac{1}{\sqrt{\Gamma}(1-\alpha)} \int_0^{1/A} \rho^{-1/2} (1-\rho)^{x-1} d\rho
\] (28)

Furthermore, to explore the behaviour of the phreatic curve near the exit point, one may introduce the appropriate value of $A$ in equation (28). It is easily shown that at point 1 the slope of the tangent to the phreatic curve is given by
\[
\frac{dy}{dx} \bigg|_{x=x_1} = -\frac{1}{\sqrt{\Gamma}(1-\alpha)} \int_0^{1/A} \rho^{-1/2} (1-\rho)^{x-1} dx
\]
\[
= -\frac{1}{\sqrt{\Gamma}(1-\alpha)} \frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)} \frac{\sqrt{(\pi)\Gamma(\alpha)}}{\Gamma(1+\alpha)} = -\frac{\pi}{\sin(\pi\alpha)} = -\cot(\pi\alpha)
\] (29)

This implies that at point 1 the phreatic curve is tangential to the sloping face of the medium, as could be predicted by the theory.

**Approximation of the parameter $\psi_0$**

At very large distances from the sloping face,
\[
A = (1 - 2\alpha) \left(1 - \frac{x}{\psi_0}\right) \gg 1
\] (30)

Hence the following approximation holds:
\[
2\sqrt{(A)} \left(1 - \frac{1}{A}\right)^{x} = 2\sqrt{(A)} \left(1 - \frac{\alpha}{A}\right) + O\left(\frac{1}{A^{3/2}}\right)
\] (31)

The integral in (25) can be approximated as
\[
\int_0^{1/A} \rho^{-1/2} (1-\rho)^{x-1} d\rho = \int_0^{1/A} \left[\rho^{-1/2} + (1-\alpha)\rho^{1/2}\right] d\rho = \frac{2}{\sqrt{(A)}} + \frac{2(1-\alpha)}{3A} + O(A^{-7/2})
\] (32)

Introducing these two approximations into (25), one obtains
\[
\sqrt{(\pi)} \frac{y}{\psi_0} \frac{\Gamma(\frac{1}{4} - \alpha)}{\Gamma(1-\alpha)} = 2\sqrt{(\pi)} + 2(\frac{1}{3}) + O\left(\frac{1}{A^{3/2}}\right)
\] (33a)

If it is known that $x = -L$ and $y = H$ at a distant point on the phreatic curve, then (33a) becomes
\[
\sqrt{(\pi)} \frac{H}{4L} \frac{1 + 2a}{\psi_0} \frac{\Gamma(\frac{1}{4} - \alpha)}{\Gamma(1-\alpha)} = 1 + \frac{1}{2} + \frac{2}{3} + O(\rho^3)
\] (33b)

where the following notation is used:
\[
\rho = \sqrt{\left(\frac{\psi_0}{1 - 2\alpha}L\right)}; \quad \eta = \frac{H}{L}; \quad A = (1 - 2\alpha) \left(1 + \frac{L}{\psi_0}\right) = 1 - 2\alpha + \frac{1}{\rho^2}
\] (34)

Next $\rho$ is sought in the form
\[
\rho = \rho_0 + \rho_1 \eta + \rho_2 \eta^2 + \rho_3 \eta^3 + \rho_4 \eta^4 + O(\eta^5)
\] (35a)

Substitution of equations (34) and (35a) in equation (33b) yields the values of the coefficients $\rho_\kappa$ ($\kappa = 0, 1, 2, 3, 4$), and equation (35a) becomes
\[
\rho = \rho_1 \eta + \frac{4\alpha - 1}{6} \rho_1^3 \eta^3 + O(\eta^5)
\] (35b)
where

\[ \rho_1 = \frac{\sqrt{\pi} \Gamma\left(\frac{1}{2} - \alpha\right)}{4 \Gamma(1 - \alpha)} \]

By squaring equation (35b), one obtains

\[ \psi_0 = (1 - 2\alpha)L \left( C^2 + \frac{4\alpha - 1}{3} C^4 + O(C^6) \right) \tag{36} \]

where

\[ C = \rho_1 \eta = \frac{\sqrt{\pi} \Gamma\left(\frac{1}{2} - \alpha\right) H}{4 \Gamma(1 - \alpha) L} \]

**ASYMPTOTIC EXPANSIONS**

**Far-field expansions**

It is also possible to express the solutions presented by equation (21) in series form as

\[ \frac{\omega}{\psi_0} = \frac{2}{\sqrt{\pi} \Gamma\left(\frac{3}{2} - \alpha\right)} \left( \sqrt{(u)} \Gamma(1 - \alpha) - \sum_{n=1}^{\infty} \frac{\Gamma(n + 1 - \alpha)}{(4n^2 - 1)n!} u^{1/2 - n} \right) \tag{37} \]

since

\[ u \left( 1 - \frac{1}{u} \right)^\alpha = -\frac{\alpha}{\Gamma(1 - \alpha)} \sum_{n=0}^{\infty} \frac{\Gamma(n - \alpha)}{n!} \frac{1}{u^{\alpha - 1}} \tag{38} \]

and

\[ F\left(\frac{1}{2}; 1 - \alpha; \frac{3}{2}; 1/u\right) = \frac{1}{\Gamma(1 - \alpha)} \sum_{n=0}^{\infty} \frac{\Gamma(n + 1 - \alpha)}{(2n + 1)n!} \frac{1}{u^n} \tag{39} \]

Again, including only a few terms of the series, the functions shown in equations (38) and (39) can be expanded as

\[ (u + 2\alpha - 1)F\left(\frac{1}{2}; 1 - \alpha; \frac{3}{2}; 1/u\right) = u + \frac{5\alpha - 2}{3} + \frac{(1 - \alpha)(17\alpha - 4)}{30u} + \ldots \tag{40} \]

\[ u \left( 1 - \frac{1}{u} \right)^\alpha = u - \alpha + \frac{\alpha(\alpha - 1)}{2u} + \ldots \tag{41} \]

Hence, including only a few terms, the solution given by equation (37) can be expressed as

\[ \frac{\omega}{\psi_0} = \frac{2}{\sqrt{\pi} \Gamma\left(\frac{3}{2} - \alpha\right)} \left( \sqrt{(u)} + \frac{\alpha - 1}{3\sqrt{(u)}} + \ldots \right) \tag{42} \]

Setting

\[ \beta = \frac{\sqrt{\pi} \Gamma\left(\frac{3}{2} - \alpha\right) \omega}{4 \Gamma(1 - \alpha) \psi_0}, \]

and \( \lambda = \sqrt{(u)} \), equation (42) becomes

\[ 2\beta = \lambda + \frac{\alpha - 1}{3\lambda} + \ldots \tag{43} \]
Equation (43) gives an approximate equation

$$\lambda^2 - 2\beta \lambda + \frac{\alpha - 1}{3} + \ldots = 0$$  \tag{44}$$

which can be solved to obtain the approximate solution to the problem as

$$u \sim \frac{\pi}{4} \left( \frac{\Gamma(\frac{1}{2} - \alpha)}{\Gamma(1 - \alpha) \psi_o} \right)^2 + \frac{2}{3}(1 - \alpha)$$  \tag{45}$$

Equation (45) will be accurate enough for distant points where the absolute value of $u$ is greater than unity. From the definition of $u$ the approximate solutions can be expressed as

$$\frac{\pi}{\psi_o} \sim \frac{4\alpha - 1}{3(2\alpha - 1)} \frac{\psi}{\psi_o} - \frac{\mu^2 - \psi^2}{\psi_o^2}$$  \tag{46}$$

Equating the real and imaginary terms, the approximate solutions are obtained as

$$\frac{x}{\psi_o} \sim \frac{4\alpha - 1}{3(2\alpha - 1)} \frac{\psi}{\psi_o} - \frac{\mu \phi}{\psi_o^2}$$  \tag{47}$$

$$\frac{y}{\psi_o} \sim \frac{\phi}{\psi_o} - 2\mu \frac{\phi \psi}{\psi_o^2}$$  \tag{48}$$

where

$$\mu = \frac{\pi}{16} \frac{(1 - 2\alpha) \Gamma^2(1/2 - \alpha)}{\Gamma(1 - \alpha)}$$

It is possible to obtain approximate equations of the distant equipotential lines and streamlines from equations (47) and (48) by setting $\phi = \phi \psi_o$ and $\psi = \psi \psi_o$ respectively as the parabolas

$$\frac{x}{\psi_o} = \frac{4\alpha - 1}{3(2\alpha - 1)} \frac{\psi}{\psi_o} - \frac{(y/\psi_o)^2 - \phi^2}{4\mu \phi^2}$$  \tag{49}$$

$$\frac{x}{\psi_o} = \frac{4\alpha - 1}{3(2\alpha - 1)} - \psi^2 - \frac{\mu (y/\psi_o)^2}{1 - 2\mu \psi^2}$$  \tag{50}$$

For distant interior points $u$ can be defined as

$$u = V \epsilon^4, \quad V \gg 1, \quad 0 \leq \theta \leq \pi$$

Hence from equation (37) and the definition of $u$ the asymptotic expansions for the farfield can be expressed as

$$\frac{\phi}{\psi_o} = \frac{-2}{\sqrt{\pi} \Gamma(\frac{3}{2} - \alpha)} \sum_{n=0}^{\infty} \frac{\Gamma(n + 1 - \alpha)}{(4n^2 - 1)n!} V^{1/2 - n} \cos([n - \frac{1}{2}]\theta)$$  \tag{51}$$

$$\frac{\psi}{\psi_o} = \frac{2}{\sqrt{\pi} \Gamma(\frac{1}{2} - \alpha)} \sum_{n=0}^{\infty} \frac{\Gamma(n + 1 - \alpha)}{(4n^2 - 1)n!} V^{1/2 - n} \sin([n - \frac{1}{2}]\theta)$$  \tag{52}$$

$$\frac{x}{\psi_o} = 1 - \frac{\psi}{\psi_o} - \frac{V \cos(\theta)}{1 - 2\alpha}$$  \tag{53}$$

$$\frac{y}{\psi_o} = \frac{\phi}{\psi_o} - \frac{V \sin(\theta)}{1 - 2\alpha}$$  \tag{54}$$
Expansions near point 0

The solutions given by equation (17) can be recast in series form as

$$\frac{z}{\psi_0} = \frac{i e^{i\alpha \pi}}{2 \sqrt{(n) \Gamma \left(\frac{3}{2} - \alpha\right)}} \sum_{n=0}^{\infty} \frac{\Gamma(n + 1 - \alpha) u^{n + 3/2 - \alpha}}{(n + 1 - \alpha) (n + 3/2 - \alpha)}$$

For the nearfield, i.e., for points around point 0, the parameter $u$ can be written as

$$u = \varepsilon e^{i\theta}, \quad \varepsilon < 1, \quad 0 \leq \theta \leq \pi$$

Substituting the near-field value of $u$ into equation (55), the near-field expansions may be written as

$$\frac{x}{\psi_0} = \frac{-1}{2 \sqrt{(n) \Gamma \left(\frac{3}{2} - \alpha\right)}} \sum_{n=0}^{\infty} \frac{\Gamma(n + 1 - \alpha) u^{n + 3/2 - \alpha} \sin[\pi \alpha + (n + 3/2 - \alpha)]}{(n + 1 - \alpha) (n + 3/2 - \alpha)}$$

$$\frac{y}{\psi_0} = \frac{1}{2 \sqrt{(n) \Gamma \left(\frac{3}{2} - \alpha\right)}} \sum_{n=0}^{\infty} \frac{\Gamma(n + 1 - \alpha) u^{n + 3/2 - \alpha} \cos[\pi \alpha + (n + 3/2 - \alpha)]}{(n + 1 - \alpha) (n + 3/2 - \alpha)}$$

$$\phi = \frac{y}{\psi_0} = \frac{\varepsilon \sin(\theta)}{1 - 2\alpha}$$

$$\psi = \frac{x}{\psi_0} = \frac{\varepsilon \cos(\theta)}{1 - 2\alpha}$$

It is useful to obtain approximations of the normal derivatives on the segments 01 and 02 around point 0. After some manipulations it can be shown that these normal derivatives are approximately given by

$$\left. \frac{\partial \phi}{\partial n} \right|_{\phi=01} \sim \sin(\pi \alpha) - \Delta \left( \frac{r}{\psi_0} \right)^n$$

$$\left. \frac{\partial \phi}{\partial n} \right|_{\phi=02} \sim 1 - \Delta \left( \frac{x}{\psi_0} \right)^n$$

where

$$\Delta = \pi^\nu \left( \frac{\eta}{2\nu^2} \right)^n \left( \frac{\Gamma \left(\frac{3}{2} - \alpha\right)}{\Gamma(1 - \alpha)} \right)^{2\nu}$$

$$\eta = (2\alpha - 1)/(3 - 2\alpha)$$

$$\nu = 1/(3 - 2\alpha)$$

$$r = y/\cos(\pi \alpha)$$

Expansions around point 1

The equations developed in the previous subsections should be useful for computation of the solutions either far from or near to point 0. However, they fail to provide good approximations for points close to point 1. To obtain a convenient formula for points around point 1, one needs to transform (20) in which $I_5$ is given by (82). After substituting $r - 1 = \rho$ and $u - 1 = \nu$, equation (20) becomes

$$\omega = - U \cot(\pi \alpha) - \frac{\psi_0}{2 \sqrt{(n) \Gamma \left(\frac{3}{2} - \alpha\right)}} \int_0^{\infty} (\nu - \rho)(1 + \rho)^{-1/2 - \ve} \rho^{n-1} d\rho$$
For points near point 1, \( v \ll 1 \) and \( \rho \ll 1 \). Hence the integral in (62) can be expanded in a power series in terms of \( v \) as

\[
\frac{\omega}{\psi_0} (1-2\alpha)\tan(\pi \alpha) = u + 2\alpha - 1 + \frac{1}{\sqrt{\pi}} \Gamma(\alpha) \sum_{n=0}^{\infty} \frac{(-1)^{n+1} \Gamma(n+\frac{1}{2}+\alpha)}{(n+\alpha)(n+\alpha+1)} \frac{(u-1)^{n+1}}{n!}
\]  

(63)

Equation (63) can be rewritten in closed form as

\[
\frac{\omega}{\psi_0} = \frac{u + 2\alpha - 1}{1 - 2\alpha} \cot(\pi \alpha) + \frac{\cot(\pi \alpha) \Gamma(\alpha - \frac{1}{2})}{2\sqrt{\pi}} \Gamma(2 + \alpha) \frac{(u-1)^{n+1}}{n!} F(a; \frac{1}{2} + \alpha; 2 + \alpha; 1-u)
\]  

(64)

In particular, for points on the phreatic surface near point 1, equation (64) becomes

\[
\frac{y}{x_1} = -\frac{x}{x_1} \cot(\pi \alpha) + \frac{\cot(\pi \alpha) \Gamma(\alpha - \frac{1}{2})}{\sqrt{\pi}} \Gamma(2 + \alpha) \frac{\left(\frac{x}{x_1} - 1\right)^{1+\alpha}}{2(\frac{x}{x_1} - 1)} F(a; \frac{1}{2} + \alpha; 2 + \alpha; 2\alpha(1-x/x_1))
\]  

(65)

Also, for points far from point 1, one obtains

\[
\frac{\omega}{\psi_0} = \frac{2\cot(\pi \alpha) \Gamma(\alpha - \frac{1}{2})}{\Gamma(\alpha)} \sqrt{(u-1)} F(-\frac{1}{2}; \frac{1}{2} + \alpha; \frac{3}{2}; 1/(1-u))
\]  

(66)

where \( u \) is as given by (21b).

For points on the phreatic surface far from point 1, equation (66) becomes

\[
\frac{y}{x_1} = 2 \left(\frac{2\alpha}{\pi}\right) \cot(\pi \alpha) \frac{\Gamma(\frac{1}{2} + \alpha)}{\Gamma(1 + \alpha)} \sqrt{\frac{x}{x_1} - 1} F(-\frac{1}{2}; \frac{1}{2} + \alpha; \frac{3}{2}; 1/2\alpha(1-x/x_1))
\]  

(67)

**NUMERICAL RESULTS AND CONCLUSIONS**

The solutions obtained in the previous sections consist of complicated integrals represented as special functions or series. For particular values of \( \alpha \) the calculations are easily performed to obtain numerical values of the solutions. For the purpose of this paper, calculations were

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**Figure 7.** Free surfaces in normalized co-ordinate space for \( \alpha = 0.1, 0.2, 0.3 \) and 0.4
performed on a CDC computer. The mathematical library package IMSL was used to evaluate the gamma functions and it was relatively simple to write a programme to evaluate the hypergeometric functions using this package to make the necessary calculations. The calculations were performed for various values of the slope angle $\pi \alpha$. The results of these calculations are summarized in Figures 7-10.

Figures 7 and 8 show the shape of the free surface in $\tilde{x}-\tilde{y}$ space for various values of $\alpha$. The parametric solutions given by equation (25) are presented in these figures. Figure 9 compares the shapes of the free surfaces for $\alpha$-values of 0.01 and 0 (as obtained in Reference 6). It is interesting to note that as $\alpha$ becomes very small, the free surface approaches the same shape as previously obtained for $\alpha=0$, i.e. a vertical slope, by a different approach. Finally, Figure 10 shows the complete flownet for the case of $\alpha=0.25$
It is noted in closing that the assumptions of homogeneity, isotropy and semi-infinite geometry of the flow regions were necessary to make very complex problem mathematically tractable. It is well known that, in practice, deviations from each of these assumed characteristics can strongly influence the nature of the solutions. Nevertheless, the solutions presented in this paper should be sufficiently accurate when some of these assumptions are approximately satisfied. At present, in view of the scarcity of exact solutions of relevant problems, these solutions would provide means for calibration and validation of numerical models. In particular, further investigation of the nature of the solutions in the neighbourhood of singular points may eventually lead to incorporation of improved features in the numerical models. Finally, the type of analysis presented herein is attractive not only on its own merit but also because of the potential for future extensions.

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APPENDIX I: NOTATION

\[ F \] \text{ hypergeometric function, } F(a; b; c; z) \\
\[ k \] \text{ coefficient of permeability} \\
\[ r \] \text{ distance to exit point on slope} \\
\[ W \] \text{ Kirchhoff’s function, } \frac{dz}{d\omega} \\
\[ U \] \text{ Zhukovsky’s function } z - i\omega
The integral

\[ I = \int_0^\omega d\tau \int_0^\tau \sigma^{-1/2} - \sigma^{-\sigma} (1 - \sigma)^{\sigma - 1} d\sigma \]  

appears in the solution given by equation (18). This integral and several others which stem from it are evaluated in this appendix. The integral \( I \) can be written as the sum of two integrals, \( I_1 \) and \( I_2 \).

**First integral \( I_1 \)**

\[ I_1 = \int_0^1 d\tau \int_0^\tau \sigma^{-1/2} - \sigma^{-\sigma} (1 - \sigma)^{\sigma - 1} d\sigma \]  

Integrating by parts, \( I_1 \) becomes

\[ I_1 = \int_0^1 \tau^{-1/2} - \sigma (1 - \tau)^{\sigma - 1} d\tau - \int_0^1 \tau^{1/2} - \sigma (1 - \tau)^{\sigma - 1} d\tau \]  

The two integrals which appear in (70) can be successively expressed in terms of beta and gamma functions as

\[ I_1 = B(\frac{1}{2} - \sigma, \sigma) - B(\frac{1}{2} - \sigma, \sigma) \]  

\[ I_2 = \frac{\Gamma(\sigma) \Gamma(\frac{1}{2} - \sigma) \Gamma(\frac{1}{2} - \sigma) \Gamma(\sigma) \Gamma(\frac{1}{2} - \sigma)}{\Gamma(\frac{1}{2})^2} \]  

Hence the first integral \( I_1 \) is given by

\[ I_1 = \frac{2\alpha}{\sqrt{\pi}} \Gamma(\alpha) \Gamma(\frac{1}{2} - \alpha) \]  

**Second integral \( I_2 \)**

\[ I_2 = \int_0^\omega d\tau \int_0^\tau \sigma^{-1/2} - \sigma^{-\sigma} (1 - \sigma)^{\sigma - 1} d\sigma \]
This integral can again be expressed as the sum of two integrals, \( I_3 \) and \( I_4 \):

\[
I_2 = \int_1^u \int_0^1 \sigma^{-1/2-a}(1-\sigma)^{a-1} \, d\sigma + \int_1^u \int_1^t \sigma^{-1/2-a}(1-\sigma)^{a-1} \, d\sigma
\]  

(75)

Third integral \( I_3 \)

\[
I_3 = \int_1^u \int_0^1 \sigma^{-1/2-a}(1-\sigma)^{a-1} \, d\sigma
\]  

(76)

From Reference 12 as before,

\[
I_3 = \int_1^u \frac{\Gamma(\alpha) \Gamma(\frac{1}{2} - \alpha)}{\Gamma(\frac{1}{2})} \, d\tau
\]

(77)

Hence

\[
I_3 = (u - 1) \frac{\Gamma(\alpha) \Gamma(\frac{1}{2} - \alpha)}{\sqrt{\pi}}
\]

(78)

Fourth integral \( I_4 \)

\[
I_4 = \int_1^u \int_1^t \sigma^{-1/2-a}(1-\sigma)^{a-1} \, d\sigma
\]

(79)

Also

\[
I_4 = -e^{ia} \int_1^u \int_1^t \sigma^{-1/2-a}(1-\sigma)^{a-1} \, d\sigma
\]

(80)

Fifth integral \( I_5 \)

Defining another integral as \( I_5 = -I_4 e^{-ia} \), this integral can be evaluated by parts as

\[
I_5 = u \int_1^u \tau^{-1/2-a}(1-\tau)^{a-1} \, d\tau - \int_1^u \tau^{-1/2-a}(1-\tau)^{a-1} \, d\tau
\]

(81)

In other words, it can be written as

\[
I_5 = \int_1^u (u - \tau) \tau^{-1/2-a}(1-\tau)^{a-1} \, d\tau
\]

(82)

Substituting \( \chi = 1/\tau \) and changing limits, the integral becomes

\[
I_5 = \int_1^{1/\tau} (u\chi - 1) \chi^{-3/2}(1-\chi)^{a-1} \, d\chi
\]

(83)

In other words, the integral \( I_5 \) can be written as

\[
I_5 = (u - 1) \int_1^{1/\tau} \chi^{-1/2}(1-\chi)^{a-1} \, d\chi - \int_1^{1/\tau} \chi^{-3/2}(1-\chi)^{a-1} \, d\chi
\]

(84)

It can be easily shown that after some manipulation the integral \( I_5 \) becomes

\[
I_5 = -2u^{1/2}(1-u)^a + (u + 2a - 1) \frac{\sqrt{(\pi)} \Gamma(\alpha)}{\Gamma(\frac{1}{2} + \alpha)} - (u + 2a - 1) \int_0^{1/u} \chi^{-1/2}(1-\chi)^{a-1} \, d\chi
\]

(85)
However, from Reference 12 it is known that

\[
\int_0^\chi \chi^{-1/2} (1-\chi)^{\alpha-1} d\chi = F\left(\frac{1}{2}; 1-\alpha; \frac{3}{2}; z\right)
\]

(86)

where \( F \) denotes a hypergeometric function which can be expressed as a series of the form

\[
F(\alpha; \beta; \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n n!} z^n = 1 + \frac{\alpha \cdot \beta}{\gamma} z + \frac{\alpha (\alpha + 1) \cdot \beta (\beta + 1)}{\gamma (\gamma + 1) \cdot 2} z^2 + \ldots
\]

(87)

Hence \( I_5 \) can also be written as

\[
I_5 = -2\sqrt{(u)(1-u)^a} + (u + 2a - 1) \left( \frac{\sqrt{(\pi)} \Gamma(\alpha)}{\Gamma\left(\frac{1}{2} + \alpha\right)} - F\left(\frac{1}{2}; 1-\alpha; \frac{3}{2}; 1/u\right) \right)
\]

(88)

Finally, therefore, the integral \( I \) is given by

\[
I = I_1 + I_3 - I_5 e^{i\pi a} = (u + 2a - 1) \left( \frac{\Gamma(\alpha) \Gamma\left(\frac{1}{2} - \alpha\right)}{\sqrt{(\pi)}} - \frac{\sqrt{(\pi)} e^{i\pi a} \Gamma(\alpha)}{\Gamma\left(\frac{1}{2} + \alpha\right)} + e^{i\pi a} F\left(\frac{1}{2}; 1-\alpha; \frac{3}{2}; 1/u\right) \right) + 2e^{i\pi a} \sqrt{(u)(1-u)^a}
\]

(89)

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