ANALYTICAL SOLUTION OF STEADY SEEPAGE INTO DOUBLE-WALLED COFFERDAMS

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ABSTRACT: Analytical solutions for steady seepage into long double-walled cofferdams have been obtained by using complex variable techniques. Employing the method of successive conformal mapping, the correspondence between the complex physical and potential planes has been obtained in terms of elliptic integrals of the first and the third kind. The equations depicting the rate of seepage, the distribution of hydraulic heads on the various segments of the domain boundaries, and the exit gradients at the base of the excavation have been obtained. The graphs of solutions for the rate of steady seepage and the maximum exit gradient have also been presented in this paper in the form of dimensionless plots to facilitate computation.

INTRODUCTION

Construction excavations are often supported with double-walled cofferdams. Certainly, the performance of the cofferdams will depend upon the structural adequacy of the sheet piles and the bracing systems that prevent the inward movement of the sheeting; however, the performance of the structure is no less sensitive to seepage into the excavation and the stability of the floor of the excavation. Since heaving and piping of the floor may lead to failure of an otherwise adequate cofferdam, it is crucial to estimate the rate of flow and the exit gradients accurately.

Even though it is possible to use various numerical solution approaches like finite difference, finite element, and boundary element methods, the problem of steady seepage into such cofferdams is more commonly analyzed by approximate solution methods such as flow net technique and method of fragments (Pavlovsky 1935; Polubarinova-Kochina 1962; Harr 1962). In the past, design charts for steady seepage into cofferdams has been prepared on the basis of numerical solutions (Fox and McNamee 1948; McNamee 1949; Griffiths 1984) and physical modeling (Marsland 1956). A reasonably accurate estimate of the rate of flow into the cofferdams can be obtained by any one of the approximate approaches listed previously, but large errors in the estimate of the maximum gradients are unavoidable since none of these approximations works well near singular points.

A review of the related literature shows that analytical solution for a problem of such practical interest has been sought by several researchers. So far, the available analytical solutions are limited to the subsets of the general solutions addressed in this paper. The semi-infinite domain problems, e.g., problem of seepage into a cofferdam in an infinitely deep stratum and that of seepage around single sheet pile wall, i.e., into an infinitely wide double-walled system have been developed by Harr and Deen (1961) and Harr (1962), respectively. Analytical solutions to problems of seepage into a cofferdam in a previous stratum of finite depth have also been presented by King and Cockcroft (1972) for the cases of no excavation and full exca-
vation inside the cofferdam. In this paper, the dimensionless solutions for both flow rate and maximum (and average) exit gradients of the complete problem that considers partial excavation inside the cofferdam are presented as functions of the nondimensional ratios of the geometric dimensions of the cofferdam.

**Formulation**

The geometry of a typical cofferdam is shown in Fig. 1. Usually, one of the plan dimensions is much larger than the other and under such situations the problem can be considered to be two-dimensional without any loss of accuracy. For steady seepage obeying Darcy’s law and homogeneous isotropic flow domain, the problem becomes one of solving the Laplace equations for the conjugate harmonic functions $\phi(x,y)$ and $\psi(x,y)$ in the domain.

The flow domain in the complex physical plane ($z$-plane) is shown on Fig. 1. The following symbols are used for the dimensions shown on this figure: $T =$ thickness of the stratum, $d =$ depth of excavation, $S =$ half-width of the excavation, and $L =$ additional penetration depth of sheet pile below the floor of excavation. The other dimensions, $\Delta$ and $\Gamma$, are defined as:

$$\Delta = T - d \quad \cdots \cdots \cdots \cdots \cdots (1a)$$

and

$$\Gamma = T - (d + L) \quad \cdots \cdots \cdots \cdots \cdots (1b)$$

The boundary conditions of the problem are also shown on Fig. 1. In the $z$-plane, AB and DE are equipotential lines and correspond to $\phi =$ constant. On DE, this constant is assumed to be zero so that $\phi = 0$ along this line. Along AB, instead of $\phi = -kH$, which is normally used when the y-axis is oriented downwards, $\phi = kH$ is used here; $k$ being the coefficient of permeability of the medium and $H$ being the total hydraulic head drop between AB and DE. Along the stream line BCD the value of $\psi$ is taken to be zero. Then, along AGFE the value of $\psi$ is assumed to be $q$, where $2q$ is the total flow rate per unit length of the cofferdam, because of the symmetry of the problem.

![FIG. 1. Flow Domain in Complex Physical ($z$) Plane](image-url)
ANALYTICAL SOLUTION

An exact solution to the problem described is sought by the method of successive conformal mapping. In short, the approach involves finding appropriate Schwarz-Christoffel transformations that map polygonal domains in $z$- and $\omega$-planes on the upper half of other auxiliary planes and obtaining the functional relationships between $z$ and $\omega$ from the correspondence of the domains in the auxiliary planes.

First of all, the domain on Fig. 1 can be obtained by conformal mapping of the domain on Fig. 2. This conformal mapping is given by the integral.

$$z = S - iP \int_0^w \frac{x - \beta}{x + \alpha \sqrt{x(x - \epsilon)(x - 1)}} \, dx$$  (2)

in which

$$\alpha > 0, \quad 0 < \epsilon < 1 < \beta, \quad P > 0$$  (3)

Letting

$$\kappa = \sqrt{\epsilon}$$  (4)

and

$$\kappa' = \sqrt{1 - \kappa^2}$$  (5)

one obtains the following relations from the correspondence for points E, D, C, and B in Figs. 1 and 2:

$$\frac{\Delta}{P} = \int_0^{\kappa^2} \frac{\beta - W}{W + \alpha \sqrt{W(W^2 - W)(1 - W)}} \, dW$$  (6a)

$$\frac{S}{P} = \int_{\kappa^2}^{1} \frac{\beta - W}{W + \alpha \sqrt{W(W - \kappa^2)(1 - W)}} \, dW$$  (6b)

$$\frac{L}{P} = \int_1^{\beta} \frac{\beta - W}{W + \alpha \sqrt{W(W - \kappa^2)(W - 1)}} \, dW$$  (6c)

$$\frac{d + L}{P} = \int_{\beta}^{+\infty} \frac{W - \beta}{W + \alpha \sqrt{W(W - \kappa^2)(W - 1)}} \, dW$$  (6d)

FIG. 2. Transformation on $W$-Plane

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The integrals in (6a)–(6d) are expressed by complete elliptical integrals of first and third kind using Gradshteyn and Ryzhik (1980). Hence

\[
\frac{\Delta}{2P} = \left(1 + \frac{\beta}{\alpha}\right) \Pi_0\left(-\frac{\kappa^2}{\alpha}, \kappa\right) - K(\kappa) \tag{7a}
\]

\[
\frac{S}{2P} = \frac{\alpha + \beta}{\alpha + 1} \Pi_0\left(\frac{\kappa^2}{\alpha + 1}, \kappa\right) - K(\kappa') \tag{7b}
\]

\[
\frac{L}{2P} = -\frac{\kappa^2(\alpha + \beta)}{\alpha + \kappa^2)(\alpha + 1)} \Pi\left(\frac{\sqrt{\beta + 1}}{\sqrt{\beta - \kappa^2}}, \frac{\alpha + \kappa^2}{\kappa^2}, \kappa\right) + \frac{\beta - \kappa^2}{\alpha + \kappa^2} F\left(\frac{\sqrt{\beta - 1}}{\sqrt{\beta - \kappa^2}}, \kappa\right) \tag{7c}
\]

\[
d + \frac{L}{2P} = \left(1 + \frac{\beta}{\alpha}\right) \Pi\left(\frac{1}{\sqrt{\beta}}, -\alpha, \kappa\right) - \frac{\beta}{\alpha} F\left(\frac{1}{\sqrt{\beta}}, \kappa\right) \tag{7d}
\]

where \(F(z, \kappa) = \int_0^1 \frac{dx}{\sqrt{1 - x^2}(1 - \kappa x^2)}\) is elliptical integral of the first kind; \(K(\kappa) = F(1, \kappa)\) is complete elliptical integral of the first kind; \(\Pi(z, n, \kappa) = \int_0^1 \frac{dx}{1 + n x^2 \sqrt{1 - x^2}(1 - x^2)(1 - \kappa^2 x^2)}\) is elliptical integral of the third kind; \(\Pi_0(n, \kappa) = \Pi(1, n, \kappa)\) is the complete elliptical integral of the third kind; \(\kappa\) is the modulus; and \(n\) is the parameter of the elliptical integrals.

The four equations in this system [(7)] can be reduced from two equations with two unknowns, \(\alpha\) and \(\kappa\), since the other two unknowns, \(P\) and \(\beta\), can also be expressed in terms of \(\alpha\) and \(\kappa\) as

\[
\beta = \frac{SK(\kappa) - \Delta K(\kappa')}{S \Pi_0\left(-\frac{\kappa^2}{\alpha}, \kappa\right) - \Delta \Pi_0\left(\frac{\kappa^2}{\alpha + 1}, \kappa\right)} - \alpha \tag{8}
\]

\[
P = \frac{1}{2} \frac{K(\kappa')}{\alpha + 1} \frac{\Pi_0\left(\frac{\kappa^2}{\alpha + 1}, \kappa\right) - K(\kappa)}{\alpha} \Pi_0\left(-\frac{\kappa^2}{\alpha}, \kappa\right) \tag{9}
\]

Hence, (7) is reduced to the following system of equations

\[
(d + L) \left[\left(1 + \frac{\beta}{\alpha}\right) \Pi_0\left(-\frac{\kappa^2}{\alpha}, \kappa\right) - K(\kappa)\right] + \Delta\left[\frac{\beta}{\alpha} F\left(\frac{1}{\sqrt{\beta}}, \kappa\right)\right] \tag{10a}
\]

\[
- \left(1 + \frac{\beta}{\alpha}\right) \Pi\left(\frac{1}{\sqrt{\beta}}, -\alpha, \kappa\right) = 0
\]

\[
\frac{L}{\alpha} (\alpha + \kappa^2) \left[(\alpha + \beta) \Pi_0\left(-\frac{\kappa^2}{\alpha}, \kappa\right) - \alpha K(\kappa)\right]
\]

\[
+ \Delta \left(\frac{\kappa^2}{\alpha + 1} \Pi\left(\frac{\sqrt{\beta - 1}}{\sqrt{\beta - \kappa^2}}, \frac{\alpha + \kappa^2}{\kappa^2}, \kappa\right)\right)
\]
\[ - (\beta - \kappa^2) F \left( \frac{\beta - 1}{\sqrt{\beta - \kappa^2}}, \kappa \right) = 0 \] \hspace{1cm} (10b)

where \( \beta \) is given previously by (8). The nonlinear system of equations [(10)] could eventually be solved to obtain the unknowns, \( \alpha \) and \( \kappa \). After \( \alpha \) and \( \varepsilon \) (= \( \kappa^2 \)) are found, \( \beta \) and \( P \) can be obtained from (8) and (9), respectively.

To solve the problem at hand, one really needs to transform conformally the domain on Fig. 1 onto the domain on Fig. 3 (\( \omega \)-plane) where \( \omega = \phi + i\psi \) is the complex potential. It should also be noted here that \( \phi_C \), \( \phi_F \), \( \phi_G \), and \( q \) are the unknowns of the problem. It is convenient first to consider an auxiliary function, \( \tilde{\omega} \), given by

\[ \tilde{\omega} = \frac{\omega}{q} K(m') - K(m) \] \hspace{1cm} (11)

which maps the domain on Fig. 3 onto the domain on Fig. 4. In (11), the modulus, \( m(0 < m < 1) \), is an unknown constant, and the complementary modulus, \( m' \), and \( K(m) \) are given by

\[ m' = \sqrt{1 - m^2} \] \hspace{1cm} (12)

\[ K(m) = \int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - m^2x^2)}} \] \hspace{1cm} (13)

\[ \begin{array}{c}
\text{FIG. 3. Flow Domain in Complex Potential (\( \omega \) Plane)} \\
\end{array} \]

\[ \begin{array}{c}
\text{FIG. 4. Transformation on \( \bar{\omega} \)-Plane} \\
\end{array} \]
The correspondence of points $B_3$ (Fig. 3) and $B_4$ (Fig. 4) yields the relationship for $q$

$$q = \frac{1}{2} kH \frac{K(m')}{K(m)} \quad \cdots \quad (14)$$

provided $m$ is known. Next, considering another auxiliary function, $\zeta (= \xi + i\eta)$ defined as the Jacobian elliptic sine amplitude function

$$\zeta = sn(\omega, m) \quad \cdots \quad (15)$$

it is possible to map the rectangular domain on Fig. 4 onto the upper half $\zeta$-plane as shown in Fig. 5.

Finally, the problem is completely solved by conformal mapping between the half-planes on Figs. 2 and 5. This is performed by a bilinear function. From the correspondence of points $A_2, B_2, D_2$ (Fig. 2) and $A_5, B_5, D_5$ (Fig. 5), the bilinear transformation function is obtained to be

$$W = \frac{2m + \alpha(m - 1)}{m + 1} + \frac{2(m - 1)(\alpha + 1)}{(m + 1)(\zeta - 1)} \quad \cdots \quad (16)$$

The modulus, $m$, is derived from the correspondence of points $E_2$ and $E_5$.

$$m = \frac{\alpha - \varepsilon + 2 - 2\sqrt{(1 - \varepsilon)(1 + \alpha)}}{\alpha + \varepsilon} \quad \cdots \quad (17)$$

Eqs. (11), (15), and (16) can now be combined to obtain

$$W = \frac{2m + \alpha(m - 1)}{m + 1} + 2 \left( \frac{m - 1}{m + 1} \right) \left[ \frac{\alpha + 1}{\omega K(m') - K(m), m} \right] \quad \cdots \quad (18)$$

The solution, in the form $z(\omega)$, i.e., $[x(\phi, \psi), y(\phi, \psi)]$, is given by the pair of equations [(18) and (19)] which is obtained by rearrangement of (2) as

$$z = S + iP \left[ - \int_0^w \frac{dx}{\sqrt{x(x - \varepsilon)(x - 1)}} \right] + (\alpha + \beta) \int_0^w \frac{dx}{(x + \alpha)\sqrt{x(x - \varepsilon)(x - 1)}} \quad \cdots \quad (19)$$

\[ \text{FIG. 5. Transformation on } z\text{-Plane} \]
Potentials on Various Segments of the Flow Domain

As pointed out earlier, the integrals in (19) can be expressed by elliptic integrals of first and third kind using Gradshteyn and Ryzhik (1980). In this section, exact expressions for the potentials on different parts of the domain ABCDEFGA have been obtained in terms of elliptic integrals of first and third kind. In particular, the solutions for the unknowns \( \phi_G, \phi_C, \) and \( \phi_F \) are obtained here.

It can be noted that on the flow line AGFE \((\psi = q)\), \( W \) is given by [(18)]

\[
W_{AE} = \frac{2m + \alpha(m - 1)}{m + 1} + 2\left(\frac{m - 1}{m + 1}\right) \frac{\alpha + 1}{1 + sn(\tilde{\omega}_{AE}, m)} \quad \cdots \cdots \quad (20)
\]

where \( \tilde{\omega}_{AE} = (\phi/q)K(m') - K(m) + iK(m') \). Eq. (20) can be simplified as

\[
W_{AE} = \frac{2m + \alpha(m - 1)}{m + 1} + 2m\left(\frac{1 - m}{1 + m}\right) \frac{\alpha + 1}{m + \frac{dn(u, m)}{cn(u, m)}} \quad \cdots \cdots \quad (21)
\]

in which \( u = (\phi/q)K(m') \). Then, from (19), the implicit relation between physical coordinates and the potentials on this segment can be obtained as:

1. On FE, \( x = S, \psi = q, \) and

\[
y = 2P\left[\left(1 + \frac{\beta}{\alpha}\right)\Pi\left(\frac{\sqrt{W}}{\kappa}, -\frac{\kappa^2}{\alpha}, \kappa\right) - F\left(\frac{\sqrt{W}}{\kappa}, \kappa\right)\right] \quad \cdots \cdots \quad (22)
\]

2. On AF, \( y = 0, \psi = q, \) and

\[
x = S + \frac{2P}{\alpha + \kappa^2}\left[\kappa^2\left(1 + \frac{\beta}{\alpha}\right)\Pi\left(\theta, 1 + \frac{\kappa^2}{\alpha}, \kappa'\right) + (\beta - \kappa^2)F(\theta, \kappa')\right] \quad \cdots \cdots \quad (23)
\]

in which, \( \theta = 1/\sqrt{1(\kappa^2/W)} \)

Eq. (23) provides an equation for the unknown \( W_G \), since at point G, i.e., at \( x = 0 \), (23) becomes

\[
\kappa^2\left(1 + \frac{\beta}{\alpha}\right)\Pi\left(\theta, 1 + \frac{\kappa^2}{\alpha}, \kappa'\right) + (\beta - \kappa^2)F(\theta, \kappa') = \frac{S}{2P}\left(\alpha + \kappa^2\right) \quad \cdots \cdots \quad (24)
\]

After \( \theta \) is found from (24), one obtains \( W_G \)

\[
W_G = \frac{\kappa^2\theta^2}{1 - \theta^2} \quad \cdots \cdots \quad (25)
\]

Then, to obtain \( \phi_G \), instead of using another implicit relationship, one would rather use a direct approach since the inverses of (16), (15), and (11) can be written, respectively, as

\[
\xi_G = 1 + \frac{2(m - 1)(\alpha + 1)}{(m + 1)W_G - 2m + \alpha(1 - m)} \quad \cdots \cdots \quad (26)
\]

\[
\tilde{\omega}_G = F(\xi_G, m) \quad \cdots \cdots \quad (27)
\]

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\[ \omega_G = \frac{q}{K(m')} [\omega_G + K(m)] \] .............................. (28)

Similarly, for the segment BCD, \(W\) is found from (18) to be

\[ W_{BD} = \frac{2m + \alpha(m - 1)}{1 + m} + 2 \left( \frac{1 - m}{1 + m} \right) \frac{\alpha + 1}{1 + sn(u, \ m)} \] .......................... (29)

in which \(u = K(m) - (\psi/q)K(m')\). On CD, \(x = 0, \psi = 0\), and

\[ y = \Delta - 2P \left[ \frac{\kappa^2(\alpha + \beta)}{(\alpha + 1)(\alpha + \kappa'^2)} \Pi \left( \gamma, \frac{\alpha + \kappa^2}{\kappa'^2}, \kappa \right) + \frac{\beta - \kappa^2}{\alpha + \kappa^2} F(\gamma, \kappa) \right] \] ............... (30)

in which, \(\gamma = \sqrt{1 - (\kappa'^2/W - \kappa^2)}\). On BC, \(x = 0, \psi = 0\), and

\[ y = T + P \left[ (1 + \frac{\beta}{\alpha}) \Pi \left( \frac{1}{\sqrt{W}}, \alpha, \kappa \right) - \frac{\beta}{\alpha} F \left( \frac{1}{\sqrt{W}}, \kappa \right) \right] \] ............... (31)

On DE, \(y = \Delta, \phi = 0\), and

\[ x = 2P \left[ F \left( \frac{\sqrt{1 - \frac{1}{W}}}{\kappa'}, \kappa' \right) - \frac{\alpha + \beta}{\alpha + 1} \Pi \left( \frac{\sqrt{1 - \frac{1}{W}}}{\kappa'}, \frac{\kappa'^2}{\alpha + 1}, \kappa' \right) \right] \] ............... (32)

Eq. (18), for \(W\), on DE becomes

\[ W_{DE} = \frac{2m + \alpha(m - 1)}{1 + m} + 2 \left( \frac{1 - m}{1 + m} \right) \frac{1 + \alpha}{1 + \frac{1}{dn(u, \ m')}} \] .......................... (33)

in which \(u = (\psi/q)K(m')\). Finally, on AB, \(y = T, \phi = kH\), and

\[ x = 2P \left[ \frac{\beta - 1}{1 + \alpha} F \left( \frac{1}{\sqrt{1 - \frac{1}{W}}}, \kappa' \right) - \frac{\alpha + \beta}{1 + \alpha} \Pi \left( \frac{1}{\sqrt{1 - \frac{1}{W}}}, 1 + \alpha, \kappa' \right) \right] \] ............... (34)

and \(W\) is found from

\[ W_{AB} = \frac{2m + \alpha(m - 1)}{m + 1} + 2 \left( \frac{1 - m}{1 + m} \right) \frac{\alpha + 1}{1 - \frac{1}{dn(u, \ m')}} \] .......................... (35)

in which \(u = (\psi/q)K(m')\). The remaining unknowns, \(\phi_C\) and \(\phi_F\), are found in a similar manner as in case of \(\phi_G\). From the inverses of (11), (15), and (16) one obtains

\[ \omega_C = \phi_C + iq = \frac{kH}{2} + \frac{q}{K(m')} F \left[ 1 + \frac{2(m - 1)(\alpha + 1)}{(m + 1)\beta - 2m + \alpha(1 - m)}, m \right] \] ............... (36)

\[ \omega_F = \phi_F + iq = \frac{kH}{2} + \frac{q}{K(m')} F \left[ 1 + \frac{2(m - 1)(\alpha + 1)}{\alpha(1 - m) - 2m}, m \right] \] ............... (37)

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Exit Gradients

The stability of the floor of the cofferdam against failure by heaving and piping depends on the magnitude of the seepage forces (i.e., the hydraulic gradients) acting on the soil mass at the flow boundary DE (Fig. 1). While the phenomenon of heaving, which refers to the general upheaval, relates to the average gradient, local failure by formation of boils that lead to piping relates to the maximum gradient. It is, therefore, important to find the distribution of the hydraulic gradients along the base, DE.

The exit (hydraulic) gradient, \( i_E \) at any point is given by

\[
i_E = \frac{1}{k} \frac{\partial \phi}{\partial n} \tag{38}
\]

where \( n = \) the direction normal to the flow boundary; and \( k = 1 \) the coefficient of permeability. At the base of excavation, DE, we have \( z = x + i \Delta \) and \( \omega = i \psi \) since \( y = \Delta \), and \( \phi = 0 \), and

\[
\frac{\partial \phi}{\partial n} = \frac{\partial \phi}{\partial y} \tag{39}
\]

or

\[
\frac{\partial \phi}{\partial n} = -\frac{\partial \psi}{\partial x} \tag{40}
\]

Alternatively

\[
\frac{\partial \phi}{\partial n} = -\left(\frac{dx}{dW}\right)^{-1} \tag{41}
\]

In order to find the derivatives \( dx/dW \) and \( dW/d\psi \), let us recall that, on DE, we have

\[
x + i \Delta = S + iP \left[ - \int_0^w \frac{dx}{\sqrt{x(x-\epsilon)(x-1)}} \right. \]

\[
+ \left. (\alpha + \beta) \int_0^w \frac{dx}{(x + \alpha)\sqrt{x(x-\epsilon)(x-1)}} \right] \tag{42}
\]

and \( W \) is given by (33). Hence

\[
\frac{dx}{dW} = \frac{P(\beta - W)}{(W + \alpha)\sqrt{W(W - \kappa^2)(1 - W)}} \tag{43}
\]

and

\[
\frac{dW}{d\psi} = \frac{2K(m')}{q} \left(1 + \alpha\right) \frac{(1 - m)^2}{m'} \frac{sn(u, m')cn(u, m')}{[1 + dn(u, m')]^2} \tag{44}
\]

where \( u = (\psi/q)K(m') \). The quantity \( \partial \phi/\partial n \) along DE can be found by substituting (43) and (44) into (40), and the exit gradient can be obtained from (38). Only at point D it is not possible to obtain the value of \( \partial \phi/\partial n \) in this manner since \( W \) becomes equal to 1 at this point. It is at this point, however, the maximum exit gradient is expected to occur. Therefore, we must seek the limit of \( \partial \phi/\partial n \) at point M (on DE) as M approaches the point
D. Let $b = (\psi/q)K(m')$, and $W = 1 - \xi$. We can note that as $M$ approaches $D$, $\psi \rightarrow 0$, $W \rightarrow 1$, $\xi \rightarrow 0$, and $b \rightarrow 0$. Then $snb \sim b$, $cnb \sim 1 - b^2/2$ and $dnb = dn(b, m') = 1 - m'^2b^2/2$. Hence, from (42) we get

$$1 - \xi \sim \frac{2 + \alpha - \alpha m}{m + 1} + 2\left(\frac{m - 1}{m + 1}\right) = \frac{1 + \alpha}{2m'^2b^2}$$ ................................................................. (45)

Some simple manipulation of (45) yields

$$\xi \sim \frac{1}{4} (1 + \alpha)(1 - m)^2b^2$$ ................................................................. (46)

From (43), as $W$ approaches 1, one obtains

$$\frac{dx}{dW} \sim \frac{P(\beta - 1)}{(1 + \alpha)\sqrt{(1 - \kappa^2)\xi}}$$ ................................................................. (47)

Also from (44) the limit of the other derivatives is obtained as

$$\frac{dW}{d\psi} \sim \frac{2K(m')}{q} \frac{(1 + \alpha)(1 - m)^2}{m'}. \frac{b}{4}$$ ................................................................. (48)

Combining (46)–(48), we get

$$\frac{dx}{dW} \frac{dW}{d\psi} \sim \frac{K(m')}{q} \frac{P(\beta - 1)(1 - m)}{m'\kappa'\sqrt{1 + \alpha}}$$ ................................................................. (49)

Hence

$$\lim_{M \to D} M \frac{\partial \phi}{\partial n} = q \frac{\kappa'\sqrt{1 + \alpha}}{K(m')} \frac{\sqrt{1 + m}}{P(\beta - 1) \sqrt{1 - m}}$$ ................................................................. (50)

Now, we recall from (14) that

$$\frac{q}{K(m')} = \frac{kH}{2K(m)}$$ ................................................................. (51)

Therefore, the maximum exit gradient is obtained as

$$(i_{E})_{\text{max}} = \frac{H}{2K(m)} \frac{\kappa'}{P(\beta - 1)} \frac{\sqrt{1 + \alpha}}{\sqrt{1 - m}}$$ ................................................................. (52)

It is also possible to obtain an estimate of the average value of the exit gradient by considering flow rate, $q$, to occur uniformly across the width $S$ of the base. Hence, from (14) we may write

$$(i_{E})_{\text{average}} = \frac{1}{2} \frac{H}{S} \frac{K(m')}{K(m)}$$ ................................................................. (53)

It should be noted that the maximum exit gradient can be used to check the possibility of piping, and the average exit gradient may be used to ensure safety against basal heave even though the conventional approach (Harr 1962) for evaluating safety against basal heave is somewhat different.
NUMERICAL RESULTS

It is worthwhile mentioning here that the calculation of the solutions presented are perfectly straightforward. The notations and Fortran subroutines for computing the elliptic integrals of the first and third kind and the Jacobian elliptic functions are readily available in textbooks (Press et. al. 1986) and in standard mathematical packages (ISML Library 1980).

For this study, a package [International Standard Mathematical Library (IMSL)] subroutine was used to solve the system of nonlinear equations [(10)] to obtain the unknowns $\alpha$, $\varepsilon$ ($= \kappa^2$), $\beta$, and $P$ for reasonable ranges of the problem dimensions. While the normalized physical dimension, $\Delta/d$ was varied through the values of 1.0, 2.5, 5.0, and 50.0, the other two normalized dimensions, $S/d$ and $L/d$, were chosen to lie within practical ranges of values depending on the value of $\Delta/d$. The nonlinear set of equations [(10a) and (10b)] was solved for the roots $\alpha$ and $\varepsilon$. The variations of the quantities $\alpha$ and $\varepsilon$ with the normalized geometric dimensions are shown in Figs. 6 and 7, respectively. Fig. 6 is presented as a semilogarithmic plot because the values of $\alpha$ varies over several orders of magnitude.

Clearly, the modulus, $m$, is also dependent on the physical dimensions of the problem and can be obtained from (17). The variations of modulus, $m$, with the dimensionless physical parameters is presented in Fig. 8. As given by (14), the normalized flow rate, $Q_n$ ($= q/kH$) is a function only of modulus, $m$. Fig. 9 shows the plot of normalized flow rate, $Q_n$, against modulus, $m$.

Finally, the normalized potential, $\bar{\phi}_C$ ($= \phi_C/kH$) at the tip of the sheetpile [given by (36)] and the maximum exit gradients [given by (52)] were calculated for the ranges of physical dimensions of the problem. Fig. 10 shows the variation of $\bar{\phi}_C$ with the dimensionless physical parameters, and Fig. 11 presents the dimensionless ratio $d/H(i_E)_{\text{max}}$ plotted against modulus, $m$.

**FIG. 6.** Variation of Parameter, $\alpha$, with Normalized Geometric Dimensions
FIG. 7. Variation of Parameter, $\varepsilon$, with Normalized Geometric Dimensions

FIG. 8. Variation of Modulus, $m$, with Normalized Geometric Dimensions

These graphical representations of the solutions in Figs. 6–11 can be employed for design analyses to obtain required penetration depth, $L$, which may satisfy a given set of design safety criteria. Figs. 10 and 11 can be used for computing the factors of safety against heaving and piping, respectively.
FIG. 9. Variation of Normalized Flow Rate, $Q_n$, with Modulus, $m$

FIG. 10. Variation of Normalized Potential at Tip of Sheetpile, $\Phi_c$, with Normalized Geometric Dimensions

EXAMPLE PROBLEM

To assess the present solutions and to illustrate the procedure for solving design problems with the aid of the solutions presented in the earlier section, an example problem is analyzed for both seepage and stability. The example is taken from Harr and Deen (1961) mainly in order that the answers can
FIG. 11. Graph of Normalized Maximum Exit Gradient, $d/H(i_E)_{\text{max}}$ as Function of Modulus, $m$ and Normalized Geometric Dimensions

be compared with those obtained from the previous solutions (Harr and Deen 1961; and King and Cockroft, 1972).

For the example problem, let it be given that $S = 5$ ft (1.52 m), $d = 10$ ft (3.05 m), $H = 15$ ft (4.57 m), $k = 1$ ft/day ($4 \times 10^{-4}$ cm/s), $\gamma m$ (saturated) = 116.4 pcf (18.3 kN/m$^3$). Let the additional penetration depth $L = 5$ (1.52 m) ft (after Harr and Deen 1961) and the total thickness of the stratum $T = 20$ ft (6.1 m) be chosen. For this problem, $\Delta/d = 1.0$, $S/d = 0.5$, $L/d = 0.5$, and $d/H = 0.67$. From Figs. 8–11, the values of the modulus, $m$, normalized flow rate, $Q_n$, normalized potential at the tip of sheetpile, $\Phi_c$, and the normalized exit gradient $d/H(i_E)_{\text{max}}$ are, respectively, obtained as: $m = 0.925$, $Q_n = 0.325$, $\Phi_c = 0.487$, and $d/H(i_E)_{\text{max}} = 0.24$. Hence, we may obtain: (1) Total flow rate per foot of excavation = $2 \times 15 \times 0.325 = 9.8$ cu ft/day ($0.119$ m$^3$/s); (2) maximum exit gradient, $(i_E)_{\text{max}} = 0.36$; (3) factor of safety against piping, $F_p = [i_{cr}/(i_E)_{\text{max}}] = 2.4$, since $i_{cr} = \gamma_c/\gamma_w = 54/62.4 = 0.865$; (4) head at the pile tip, $h_c = 15 \times 0.487 = 7.3$ ft (2.22 m) (5) average hydraulic gradient at the $D/S$ end of pile, $i_p = h_c/L = 1.46$; and (6) factor of safety against basal heave, $F_H = [i_{cr}/(i_E)_{\text{avg}}] = 0.89^*$, since $(i_E)_{\text{avg}} = q/kS = 4.88/5 = 0.976$ (alternatively, $F_H = i_{cr}/i_p = 0.59$). For infinite total thickness of the stratum, the other problem parameters being the same, Harr and Deen (1961) obtains total flow rate, factor of safety against piping, and factor of safety against basal heave as 13.5 cu ft/day/ft (0.164 m$^3$/s/m), 0.72, and 0.53, respectively. As expected, the semi-infinite domain solutions yield similar but conservative results. For this problem, the two approximate solutions presented by King and Cockroft (1972) give very different flow rates; while the assumption of no excavation yields total flow rate as 12 cu ft/day/ft (0.145 m$^3$/s/m), full excavation assumption yields total flow rate of 26.1 cu ft/day/ft (0.316 m$^3$/s/m). The average exit gradient adjacent to the pile for the assumption of no excavation
is obtained as 0.8 as opposed to maximum exit gradient, \((i_E)_{\text{max}} = 0.36\) given by this study.

**Conclusions**

In this paper, the problem of steady flow into double-walled cofferdams founded in finite depth of homogeneous previous medium has been addressed. For most cofferdams used in civil engineering projects, the plan dimensions are such that their length is large compared to the width. As such, a two-dimensional analysis is appropriate for long cofferdams used for the construction of dams, bridge piers, and quay walls. Using complex variable techniques, analytical solutions for the rate of steady seepage, the distribution of hydraulic heads on the various segments of the domain boundaries, and the exit gradients at the base of the excavation have been obtained. The solutions for the rate of seepage and the maximum exit gradient have been presented in this paper in the form of dimensionless plots to facilitate computation. Some comparisons of the present solutions with other existing analytical solutions show that the results obtained from those solutions are somewhat conservative.

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**Appendix. References**


