Chapter 5
Time Value of Money 2:
Analyzing Annuity Cash Flows

1. Future Value of Multiple Cash Flows
2. Future Value of an Annuity
3. Present Value of an Annuity
4. Perpetuities
5. Other Compounding Periods
6. Effective Annual Rates (EAR)
7. Amortized Loans

Future Value of Multiple Cash Flows

\[ i = 10\% \]

\[ 
\begin{array}{ccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\$1,000 & $500 & $500 & $500 & $500 & $750 \\
\end{array}
\]

\[ 
\begin{array}{ccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\$1,000 & $500 & $500 & $500 & $500 & $750 \\
\end{array}
\]

\[ \text{FV}_6 = \]

Future Value of an Ordinary Annuity

**Annuity** - A series of payments of a fixed amount for a specified number of periods of equal length

**Ordinary Annuity** - An annuity where the first payment occurs at the end of period

ex. Future Value of an Ordinary Annuity - $1,000/year, for 5 years at 12%

\[ 
\begin{array}{ccccccc}
0 & 1 & 2 & 3 & 4 & 5 \\
\$1,000 & $1,000 & $1,000 & $1,000 & $1,000 \\
\end{array}
\]

\[ \text{FV}_5 = \]
How to Solve??

Take FV of each payment at 12%.

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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<tbody>
<tr>
<td>$1,000</td>
<td>$1,000</td>
<td>$1,000</td>
<td>$1,000</td>
<td>$1,000</td>
<td>$1,000</td>
</tr>
<tr>
<td>$1,120.00</td>
<td>$1,254.40</td>
<td>$1,404.93</td>
<td>$1,573.52</td>
<td>$6,352.85</td>
<td></td>
</tr>
</tbody>
</table>

(1) Equation

\[
FV = \frac{PMT \times (1 + i)^N - 1}{i} = $1,000 \times \frac{(1.12)^5 - 1}{0.12} = $1,000 \times 6.3528 = $6,352.80
\]

(2) Calculator  PMT=1000, N=5, I=12%,  cpt FV = 6,352.85

Future Value of an Annuity Due

Annuity Due - An annuity where the first payment is at the beginning of the period

ex.  annuity due = $1,000/ year for 5 years at 12%

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1,000</td>
<td>$1,000</td>
<td>$1,000</td>
<td>$1,000</td>
<td>$1,000</td>
<td>$1,120.00</td>
</tr>
<tr>
<td>$1,120.00</td>
<td>$1,254.40</td>
<td>$1,404.93</td>
<td>$1,573.52</td>
<td>$1,762.34</td>
<td>$7,115.40</td>
</tr>
</tbody>
</table>
(1) Equation

\[ FV = PMT \times \frac{(1+i)^N - 1}{i} \times (1+i) = 1000 \times \frac{(1.12)^5 - 1}{0.12} \]

\[ = 1,000 \times 6.3528 \times 1.12 = 7,115.40 \]

(2) Calculator

BEG mode

PMT=1000, I=12%, N=5,

cpt FV = 7,115.19

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Present Value of an Ordinary Annuity

ex. Present Value of an Ordinary Annuity - $1,000/year, for 5 years at 12%

(1) Equation:

\[ PV = PMT \times \left[ \frac{1}{i} - \frac{1}{i(1+i)^N} \right] \]

\[ = 1,000 \times \left[ \frac{1}{0.12} - \frac{1}{0.12(1+0.12)^5} \right] \]

\[ = 3,604.77 \]

(2) Calculator:

PMT=1,000, N=5, I=12%, cpt PV=3,604.78
Present Value of an Annuity Due

ex. annuity due = $1,000/ year for 5 years at 12%

<table>
<thead>
<tr>
<th>Year</th>
<th>Payment</th>
<th>Discount Factor</th>
<th>Present Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$1,000</td>
<td>1</td>
<td>$1,000</td>
</tr>
<tr>
<td>1</td>
<td>$1,000</td>
<td>0.89286</td>
<td>$892.86</td>
</tr>
<tr>
<td>2</td>
<td>$1,000</td>
<td>0.79719</td>
<td>$797.19</td>
</tr>
<tr>
<td>3</td>
<td>$1,000</td>
<td>0.71178</td>
<td>$711.78</td>
</tr>
<tr>
<td>4</td>
<td>$1,000</td>
<td>0.63552</td>
<td>$635.52</td>
</tr>
<tr>
<td>5</td>
<td>$1,000</td>
<td>0.56743</td>
<td>$567.43</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td><strong>$4,037.35</strong></td>
</tr>
</tbody>
</table>

(1) Equation:

\[ PV = PMT \left[ \frac{1}{i} - \frac{1}{i(1+i)^N} \right] \ast [1+i] \]

\[ = $1,000 \left[ \frac{1}{.12} - \frac{1}{.12(1+.12)^5} \right] \ast [1+.12] \]

\[ = $4,037.34 \]

(2) Calculator:

Calculator: (Begin Mode)

PMT=$1,000, N=5, I=12\%, \text{ cpt } PV = $4,037.35

---

Assume you start investing for your retirement by opening a Roth IRA and depositing money into a mutual fund on a yearly basis. These deposits consist of $3,000 per year and are in the form of a 40-year annuity due. Assume this fund earns 11% per year over these 40 years. How much money will be in your retirement account the day you retire? When you retire in year 40 you move your IRA nest egg into a safer account earning 5% per year. Assume you wish to withdraw an equal annual amount for 25 years as an ordinary annuity until all the money is gone. How much can you withdrawal every year?
You decide that when you retire 50 years from now, you will need $200,000 a year to live comfortably for the next 20 years (you receive these payments starting 51 years from now). If money is deposited into an account with a contract rate of interest of 10 percent, how much will you need to save every year (deposit in an account) for the next 50 years (assume you make 50 payments). Assume the first savings deposit starts today and money remains in this account paying the same interest rate until all funds are paid out.

Perpetuities - Infinite series of payments. An annuity that goes on forever.

\[ PV = \frac{PMT}{i} \]

ex. A bond is contracted to make a $90 payment per year, the first payment is one year from today, there is no maturity date, and the market rate of interest is 10%. What is the PV?

\[ PV = \frac{PMT}{i} = \frac{90}{.10} = 900 \]

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Perpetuities - Infinite series of payments. An annuity that goes on forever.

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\[ PV = \frac{PMT}{i} = \frac{90}{.10} = 900 \]
What is the value today of the following stream of cash flows (4 year annuity starting year 7) at a discount rate of 9%?

What is the value of these cash flows in year 6? Year 7?, Year 10?, Year 11?

Other Compounding Periods

ex. 10% yearly vs. 10% semi-annually
How to Solve?

Formula:

$$FV_N = PV \left(1 + \frac{i}{m}\right)^{mN}$$

$$FV_1 = $100 \left(1 + \frac{.10}{2}\right)^2 = $110.25$$

Note: I = (I / m), N = (N * m)

PV = 100,
I = 10% / 2 = 5%,
N = 1 * 2 = 2,
cpt FV = $110.25

Effective Annual Rates (EAR)

$$EAR = \left[\left(1 + \frac{APR}{m}\right)^m - 1\right]$$

Note: APR (Annual Percentage Rate) = i_{nom}

Ex. APR = 10%

What is the EAR?

- annual (m=1) = 10.00%
- semi-annual (m=2) = 10.25%
- quarterly (m=4) = 10.3813%
- monthly (m=12) = 10.4713%
- daily (m=365) = 10.5156%
You see an advertisement in Money magazine from an investment company that offers an account paying a nominal interest rate of 11%. What is the effective annual rate (EAR) for the following compounding periods? The EAR of 11% compounded semi-annually? The EAR of 11% compounded monthly? If you deposit $10,000 in this account and earn 11% compounded quarterly, what is the future value in 4½ years?

Amortized Loans

Loan that is paid off in equal payments over a set period of time.

Define:

\[
PV = \text{loan amount} \\
PMT_1 = PMT_2 = \ldots = PMT_N \\
N = \text{number of payments} \\
I = \text{given} \\
FV_N = 0
\]

ex. Home mortgage: 30 year, $300,000, 6%, paid monthly

What is the yearly payment? PMT = $21,794.67

What is the monthly payment? PMT = $1,798.65 (not $21,794.67 / 12 = $1,816.22)

<table>
<thead>
<tr>
<th>Month</th>
<th>Begin Balance</th>
<th>Payment (P&amp;I)</th>
<th>Principal</th>
<th>Interest</th>
<th>End Balance</th>
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<tbody>
<tr>
<td>1</td>
<td>$300,000.00</td>
<td>$1,798.65</td>
<td>$298.65</td>
<td>$1,500.00</td>
<td>$299,701.35</td>
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<tr>
<td>2</td>
<td>$299,701.35</td>
<td>$1,798.65</td>
<td>$300.14</td>
<td>$1,498.51</td>
<td>$299,401.20</td>
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<tr>
<td>...</td>
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<tr>
<td>359</td>
<td>$3,570.50</td>
<td>$1,798.65</td>
<td>$1,780.80</td>
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<tr>
<td>360</td>
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<td>$1,798.65</td>
<td>$1,789.70</td>
<td>$8.95</td>
<td>-0-</td>
</tr>
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</table>
ex. 30-year Mortgage vs. 15-year Mortgage

Loan amount = $300,000 (monthly)
I = 6%
N = 30 vs. N = 15

PMT$_{30}$ = $1,798.65  PMT$_{15}$ = $2,531.57

<table>
<thead>
<tr>
<th>Month</th>
<th>30-year PMT</th>
<th>15-year PMT</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 month</td>
<td>I = 1,500</td>
<td>I = 1,500</td>
</tr>
<tr>
<td></td>
<td>P = 298.65</td>
<td>P = 1,031.57</td>
</tr>
<tr>
<td>2 month</td>
<td>I = 1,498.51</td>
<td>I = 1,487.34</td>
</tr>
</tbody>
</table>

Balance

5 years $279,163.07  $228,027.30
10 years $251,057.18  $130,946.90
15 years $213,146.53  $0

Does this make sense?

<table>
<thead>
<tr>
<th>Balance</th>
<th>30-year Balance</th>
<th>15-year Balance</th>
</tr>
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<tbody>
<tr>
<td>5 year</td>
<td>$279,163.07</td>
<td>$228,027.30</td>
</tr>
<tr>
<td>Payment</td>
<td>$1,798.6516</td>
<td>$2,531.5705</td>
</tr>
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</table>

N = 5, I = 6, PMT = -732.9189, m = 12, cpt FV = $51,135.77

You purchased your house 6 years ago using a 30-year, $140,000 mortgage with a contractual rate of 7% that calls for monthly payments. What is your monthly payment? What is the loan balance today after making 6 years worth of monthly payments? How much equity do you have in your house if appreciation has averaged 14% per year?
Create an amortization table for 5-year, $100,000 loan, contracted at an 8% rate and calls for annual payments.

<table>
<thead>
<tr>
<th>Year</th>
<th>Beginning Balance</th>
<th>Payment</th>
<th>Principal</th>
<th>Interest</th>
<th>Ending Balance</th>
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