Chapter 4
Time Value of Money

1. Future Value of a Lump Sum
2. Present Value of a Lump Sum
3. Future Value of Cash Flow Streams
4. Present Value of Cash Flow Streams
5. Perpetuities
6. Uneven Series of Cash Flows
7. Other Compounding Periods
8. Effective Annual Rates (EAR)
9. Amortized Loans

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Future Value - Compounding

Future Value - What is something worth in the future at a rate of interest.

ex. $1,000 today - put in bank for 1 year at 5%

\[
\begin{array}{ccc}
0 & r = 5\% & 1 \\
\end{array}
\]

\[
PV = $1,000 \\
FV_1 = \\
FV_1 = PV + \text{Interest} \\
= $1,000 + \text{Interest} \\
= $1,000 + 50 \\
= $1050
\]

\[
FV_1 = PV + (PV \times r) = PV (1 + r)
\]

How about in 2 years?

\[
\begin{array}{ccc}
0 & r = 5\% & 1 \\
\end{array}
\]

\[
\begin{array}{ccc}
 & & 2 \\
\end{array}
\]

\[
PV = $1,000 \\
$1,050 \\
FV_2 = \\
= $52.50 \\
= $1,050.00 \\
= $1,102.50
\]

Compounding = interest on interest

General form of equation = \( FV_n = PV (1 + r)^n \)
2 ways to solve

(1) equation  \[ FV_n = PV \left( 1 + r \right)^n = $1,000 \left( 1.05 \right)^2 = $1,102.50 \]

(2) Calculator  \[ PV = 1,000, \quad n = 2, \quad r = 5\% \quad FV = -1,102.50 \]

**Calculator Rules**

1. Clear Register
2. Set to 1 period per year
3. Set proper mode (Begin or End)

What is the future value (FV) of $1,000 if it is invested 20 years at 5%?

What is the future value (FV) of $1,000 if it is invested 20 years at 10%?

**Present Value - Discounting**

Present Value - What a future sum of money is worth today

\[
\begin{array}{ccc}
0 & 1 \\
\hline
r = 10\% & & \\
\end{array}
\]

\[ PV = \frac{FV_1}{(1 + r)^n} = \frac{$1,320}{(1.10)^1} = $1,200 \]

2 ways to solve

(1) equation  \[ PV = \frac{FV_1}{(1 + r)^n} = \frac{$1,320}{(1.10)^1} = $1,200 \]

(2) Calculator  \[ FV = 1,320, \quad n = 1, \quad r = 10\% \quad PV = -1,200 \]
Oscar Wright of Carson City, Nevada, while combing through his great-grandmothers trunk of remembrances, found a State of Nevada Bond that was issued the day Nevada became the 36th state on October 31, 1864. This State of Nevada bond has a face value of $100 and a stated interest rate of 22%. Why do you think the interest rate is so high? If the State of Nevada agrees to pay Oscar the face value of this bond plus interest through October 31 of last year, how much will he receive?

Example: How long will it take a $1,000 investment to double if it is invested in an account paying 8% interest?

PV = -$1,000
FV = +$2,000
r = 8%

n = 9.0065 yrs

The “Rule of 72”. A method to approximate when solving for n or r.

n = 72 / r

or

r = 72 / n

A recent advertisement in the financial section of a magazine carried the following claim: "Invest your money with us at 6 percent, compounded annually, and we guarantee to double your money sooner than you imagine." Ignoring taxes, approximately how long would it take to double your money at a nominal rate of 6 percent, compounded annually? How long if the rate is 3%? How much if the rate is 18%?

If you had invested $10,000 in Fidelity's Magellan fund when Peter Lynch became the fund manager on 5/31/77 your money would have grown to $233,139.61 by the time Lynch retired on 5/30/92. What is the annual rate of return on your investment?
**Future Value of an Ordinary Annuity**

**Annuity** - A series of payments of a fixed amount for a specified number of periods of equal length

**Ordinary Annuity** - An annuity where the first payment occurs at the end of period

ex. Future Value of an Ordinary Annuity - $1,000/year, for 5 years at 12%

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<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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<tbody>
<tr>
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</tr>
</tbody>
</table>

\[ FV = \]  

How to Solve??

Take FV of each payment at 12%.

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
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<td>$1,000.00</td>
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<tr>
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<td>$1,120.00</td>
<td>$1,254.40</td>
<td>$1,404.93</td>
<td>$1,573.52</td>
<td>$6,352.85</td>
</tr>
</tbody>
</table>

(1) Equation

\[ FV = PMT \times \left( \frac{1 + r}{r} \right)^n - 1 = \frac{1,000 \times (1.12)^5 - 1}{.12} = 6,352.80 \]

(2) Calculator PMT=1000, n=5, r=12%, FV = 6,352.85
Future Value of an Annuity Due

**Annuity Due** - An annuity where the first payment is at the beginning of the period

ex. annuity due = $1,000/year for 5 years at 12%

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<thead>
<tr>
<th>0</th>
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<td></td>
</tr>
</tbody>
</table>

$1,120.00  $1,254.40  $1,404.93  $1,573.52  $1,762.34  $7,115.40

(1) Equation

\[ FV = PMT \times \frac{(1+i)^n-1}{(1+r)} = \frac{1000 \times (1.12)^5-1}{1.12} \]

\[ = \frac{1000 \times 6.3528}{1.12} = $7,115.40 \]

(2) Calculator

BEG mode

\[ PMT = 1000, r = 12\%, n = 5, \]

\[ FV = 7,115.19 \]

Present Value of an Ordinary Annuity

ex. Present Value of an Ordinary Annuity - $1,000/year, for 5 years at 12%

(1) Equation:

\[ PV = \frac{PMT}{r} \times \left[ \frac{1 - \frac{1}{(1+r)^n}}{1 - \frac{1}{(1+r)^m}} \right] = \frac{1000}{.12} \times \left[ \frac{1 - \frac{1}{(1.12)^5}}{1 - \frac{1}{(1.12)^5}} \right] = 8,333.33 \times 0.4326 = $3,604.78 \]

(2) Calculator:

\[ PMT = 1,000, n = 5, r = 12\%, \] \[ PV = $3,604.78 \]
Present Value of an Annuity Due

ex. annuity due = $1,000/ year for 5 years at 12%

<table>
<thead>
<tr>
<th>0</th>
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<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1,000</td>
<td>$1,000</td>
<td>$1,000</td>
<td>$1,000</td>
<td>$1,000</td>
<td>$1,000</td>
</tr>
</tbody>
</table>

$892.86
$797.19
$711.78
$635.52
$\textbf{4,037.35}

(1) Equation:

\[ PV = \frac{PMT}{r} \left[ \frac{1 - \frac{1}{(1+r)^n}}{(1+r)} \right] * (1+r) = \frac{1,000}{.12} \left[ \frac{1 - \frac{1}{(1.12)^5}}{(1.12)} \right] * (1.12) \]

\[ = 8,333.33 \times 0.4326 \times 1.12 = $3,604.78 \times 1.12 = $\textbf{4,037.35} \]

(2) Calculator:

Calculator: (Begin Mode)

\[ PMT = \$1,000, \ n = 5, \ r = 12\%, \ PV = $\textbf{4,037.35} \]

Assume you start investing for your retirement by opening a Roth IRA and depositing money into a mutual fund on a yearly basis. These deposits consist of $3,000 per year and are in the form of a 40-year annuity due. Assume this fund earns 11% per year over these 40 years. How much money will be in your retirement account the day you retire? When you retire in year 40 you move your IRA nest egg into a safer account earning 5% per year. Assume you wish to withdraw an equal annual amount for 25 years as an ordinary annuity until all the money is gone. How much can you withdrawal every year?
You decide that when you retire 50 years from now, you will need $200,000 a year to live comfortably for the next 20 years (you receive these payments starting 51 years from now). If money is deposited into an account with a contract rate of interest of 10 percent, how much will you need to save every year (deposit in an account) for the next 50 years (assume you make 50 payments). Assume the first savings deposit starts today and money remains in this account paying the same interest rate until all funds are paid out.

**Perpetuities** - Infinite series of payments. An annuity that goes on forever.

\[ PV = \frac{PMT}{r} \]

ex. A bond is contracted to make a $90 payment per year, the first payment is one year from today, there is no maturity date, and the market rate of interest is 10%. What is the PV?

\[ PV = \frac{PMT}{r} = \frac{90}{0.10} = 900 \]

**Perpetuity**

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>50</th>
<th>100</th>
<th>10,000</th>
</tr>
</thead>
<tbody>
<tr>
<td>90</td>
<td>90</td>
<td>90</td>
<td>90</td>
<td>90</td>
<td>90</td>
<td>90</td>
</tr>
</tbody>
</table>

$81.82
74.38
67.62
0.77
0.0065
$900.00

**Present Value of a Growing Perpetuity**

\[ PV = \frac{CF}{r - g} \]
The Andre Agassi Foundation wishes to endow the Las Vegas Animal Shelter with an annual gift that will grow with inflation. The amount given the first year is $10,000 and inflation is expected to be 3% per year. If the discount rate is 10%, what is the present value of this endowment?

If instead inflation is estimated to be 5% per year and the discount rate remains at 10%, what is the present value of this endowment?

Uneven Cash Flow Streams

\[
\begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
\$1,000 & \$500 & \$500 & \$500 & \$500 & \$750 \\
\end{array}
\]

do in steps: Step (1)

\[
 PV_0 = \frac{\$1,000}{1.10} = 909.09
\]

Step (2)

\[
 PV_1 = \frac{\text{PMT}}{r} \times \left[ \frac{1 - \frac{1}{(1+r)^n}}{1} \right] = \frac{500}{.10} \times \left[ \frac{1 - \frac{1}{(1.10)^6}}{1} \right] = 5,000.00 \times 0.3170
\]

\[
= 5,000.00 \times 0.3170 = \$1,585.00
\]

\[
 PV_0 = \frac{\$1,585.00}{1.10} = 1,440.91
\]
Step (3)

\[ PV_0 = \frac{$750}{(1.10)^6} = $423.36 \]

Total \( PV_0 = $909.09 + $1440.91 + $423.36 = $2,773.36 \)

Assuming a discount rate of 9%, what is the value of these cash flows in year 6? Year 7?, Year 10?, Year 11?

What is the value today?

Other Compounding Periods

ex. 10% yearly vs. 10% semi-annually
How to Solve?

Formula:

$$FV_n = PV \times (1 + \frac{r}{m})^{m \times n}$$

$$FV_1 = $100 \times (1 + \frac{.10}{2})^{2 \times 1}$$

$$FV_1 = $110.25$$

Calculator:

Note: \( r = \left(\frac{r}{m}\right), n = (n \times m) \)

PV = 100,

\( r = 10\% / 2 = 5\%, \)

n = 1 * 2 = 2,

FV = $110.25

Effective Annual Rates (EAR)

$$EAR = \left(1 + \frac{r}{m}\right)^m - 1$$

Note: where \( r \) represents the Annual Percentage Rate (APR)

ex. APR = 10%

What is the EAR?

<table>
<thead>
<tr>
<th>Frequency</th>
<th>EAR</th>
</tr>
</thead>
<tbody>
<tr>
<td>annual (m=1)</td>
<td>10.00%</td>
</tr>
<tr>
<td>semi-annual (m=2)</td>
<td>10.25%</td>
</tr>
<tr>
<td>quarterly (m=4)</td>
<td>10.3813%</td>
</tr>
<tr>
<td>monthly (m=12)</td>
<td>10.4713%</td>
</tr>
<tr>
<td>daily (m=365)</td>
<td>10.5156%</td>
</tr>
</tbody>
</table>

You see an advertisement in Money magazine from an investment company that offers an account paying a nominal interest rate of 11%. What is the effective annual rate (EAR) for the following compounding periods? The EAR of 11% compounded semi-annually? The EAR of 11% compounded monthly? If you deposit $10,000 in this account and earn 11% compounded quarterly, what is the future value in 4½ years?
Amortized Loans

Loan that is paid off in equal payments over a set period of time.

Define:

- $PV =$ loan amount
- $PMT_1 = PMT_2 = \ldots = PMT_i$
- $n =$ number of payments
- $r =$ given
- $FV_i = 0$

ex. Home mortgage: 30 year, $300,000, 6\%,$ paid monthly

What is the yearly payment? $PMT = 21,794.67$

<table>
<thead>
<tr>
<th>Month</th>
<th>Begin Balance</th>
<th>Payment (P&amp;I)</th>
<th>Principal</th>
<th>Interest</th>
<th>End Balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$300,000.00</td>
<td>$1,798.65</td>
<td>$298.65</td>
<td>$1,500.00</td>
<td>$299,701.35</td>
</tr>
<tr>
<td>2</td>
<td>$299,701.35</td>
<td>$1,798.65</td>
<td>$300.14</td>
<td>$1,498.51</td>
<td>$299,401.20</td>
</tr>
<tr>
<td>...</td>
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</tr>
<tr>
<td>359</td>
<td>$3,570.50</td>
<td>$1,798.65</td>
<td>$1,780.80</td>
<td>$17.85</td>
<td>$1,789.70</td>
</tr>
<tr>
<td>360</td>
<td>$1,789.70</td>
<td>$1,798.65</td>
<td>$1,789.70</td>
<td>$8.95</td>
<td>$0</td>
</tr>
</tbody>
</table>

ex. 30-year Mortgage vs. 15-year Mortgage

Loan amount = $300,000 (monthly)
- $r = 6\%$
- $n = 30$ vs. $n = 15$

- $PMT_{30} = 1,798.65$
- $PMT_{15} = 2,531.57$

month 1
- $I = 1,500$
- $P = 298.65$
- $I = 1,500$
- $P = 1,031.57$

month 2
- $I = 1,498.51$
- $I = 1,487.34$

balance
- 5 years $279,163.07
- 10 years $251,057.18
- 15 years $213,146.53
Does this make sense?

balance
5 year $279,163.07 - $228,027.30 = $51,135.77

payment $1,798.6516 - $2,531.5705 = -$732.9189

n = 5, r = 6, PMT = -732.9189, m = 12, FV = $51,135.77

You purchased your house 6 years ago using a 30-year, $200,000 mortgage with a contractual rate of 7% that calls for monthly payments. What is your monthly payment? What is the loan balance today after making 6 years worth of monthly payments? How much equity do you have in your house if appreciation has averaged 3% per year?

Create an amortization table for 5-year, $100,000 loan, contracted at an 8% rate and calls for annual payments.

<table>
<thead>
<tr>
<th>Year</th>
<th>Beginning Balance</th>
<th>Payment</th>
<th>Principal</th>
<th>Interest</th>
<th>Ending Balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$100,000</td>
<td></td>
<td></td>
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<tr>
<td>2</td>
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<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
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