Introduction to Derivation Trees
PHIL 422
Dove/Woodbridge

1. The Tree Rules

We have nine rules:

Double Negation (DN): \( \neg \neg \phi \)
\[
\begin{array}{c}
\neg \neg \phi \\
\phi
\end{array}
\]

Conjunction: \( (\phi \land \psi) \)
\[
\begin{array}{c}
\phi \\
\psi
\end{array}
\]
Negated Conjunction: \( \neg(\phi \land \psi) \)
\[
\begin{array}{c}
\neg \phi \\
\neg \psi
\end{array}
\]

Disjunction: \( (\phi \lor \psi) \)
\[
\begin{array}{c}
\phi \\
\psi
\end{array}
\]
Negated Disjunction: \( \neg(\phi \lor \psi) \)
\[
\begin{array}{c}
\neg \phi \\
\neg \psi
\end{array}
\]

Conditional: \( (\phi \rightarrow \psi) \)
\[
\begin{array}{c}
\neg \phi \\
\psi
\end{array}
\]
Negated Conditional: \( \neg(\phi \rightarrow \psi) \)
\[
\begin{array}{c}
\phi \\
\neg \psi
\end{array}
\]

Biconditional: \( (\phi \leftrightarrow \psi) \)
\[
\begin{array}{c}
\phi \\
\neg \phi \\
\psi \\
\neg \psi
\end{array}
\]
Negated Biconditional: \( \neg(\phi \leftrightarrow \psi) \)
\[
\begin{array}{c}
\phi \\
\neg \phi
\end{array}
\]

We decompose a set of wffs, \( \Sigma \), by first listing the wffs in \( \Sigma \) vertically. We call this the trunk. Then, for each non-literal wff, apply the appropriate rule to decompose it into its simpler parts. When you apply a rule to a wff, mark it with “✓”.

A branch closes if it contains some atom, \( \chi \), and its negation, \( \neg \chi \).

We mark a closed branch with “⊗”.

A branch is complete if it is either closed OR all of its non-literal wffs have been checked.

If a branch is complete and not closed, we call it open (really, we should call it a “completed open branch”). We mark completed open branches with “☐”.

A tree is closed if all of its branches are closed.

A tree is open if at least one of its branches is a completed open branch.
2. Theorems, Anti-theorems and Neutrals

We can test whether a wff is a theorem, an anti-theorem or a neutral by using trees. To do so, first, place the wff in the trunk. Then, decompose until all branches are completed. Check for closed branches. Then, put the negation of the wff in the trunk of a new tree. Next, decompose until all branches are completed. Finally, check for closed branches.

Definitions:

- A wff is a theorem iff its tree is open and its negation’s tree is closed.
- A wff is an anti-theorem iff its tree is closed and its negation’s tree is open.
- A wff is a neutral iff both its tree and its negation’s tree are open.

3. Compatible and Incompatible Sets

We can test whether a set of wffs, $\Sigma$, is compatible using trees. To do so, first, place all of the members of $\Sigma$ in the trunk. Then decompose the trunk until all the branches are complete. Check for closed branches.

Definitions:

- A set of wffs, $\Sigma$, is compatible iff its completed tree is open.
- A set of wffs, $\Sigma$, is incompatible iff its completed tree is closed.

4. Establishments and Non-establishments

Definitions:

- A set of wffs, $\Gamma$, establishes a wff, $\phi$, iff the set of wffs $\Sigma = \Gamma \cup \{\neg \phi\}$ is incompatible.
- A set of wffs, $\Gamma$, non-establishes a wff, $\phi$, iff the set of wffs $\Sigma = \Gamma \cup \{\neg \phi\}$ is compatible.

Note: We write ‘$\Gamma$ establishes $\phi$’ as ‘$\Gamma \vdash \phi$’

5. Coupled and Uncoupled Pairs

Definitions:

- A pair of wffs, $\phi$ and $\psi$, are *coupled* iff $\phi \vdash \psi$ and $\psi \vdash \phi$ (i.e., $\phi \vdash \neg \psi$).
- A pair of wffs, $\phi$ and $\psi$, are uncoupled iff either $\phi \vdash \psi$ fails or $\psi \vdash \phi$ fails.