1. Show “(∃x(F(x) → G(x)) → ∃x(F(x) ∧ G(x)))” is contingent.

Proof: It is iff some I₁ makes the sentence true [⊕] AND some I₂ makes the sentence false [⊗].
By ⊕, I₁ either makes “∀x(F(x) → G(x))” false [⊕] OR makes “∃x(F(x) ∧ G(x))” true [⊕] [TC →]. Since either condition is sufficient, consider ⊕. By ⊕, for some α ∈ D, α does not satisfy “(F(x) → G(x))” on I₁ [TC ∀]. So, α satisfies “F(x)” on I₁ [⊗] AND α does not satisfy “G(x)” on I₁ [⊗] [TC →]. By ⊗, α ∈ “F” on I₁ [TC Atomic] AND by ⊗, α /∈ “G” on I₁ [TC Atomic]. So, take I₁ as follows: D = {1}, “F”: {1}, “G”: {}.
By ⊗, I₂ makes “∀x(F(x) → G(x))” true [⊕] AND makes “∃x(F(x) ∧ G(x))” false [⊗] [TC →]. By ⊗, for every β ∈ D, β satisfies “(F(x) → G(x))” on I₂ [⊗] [TC ∀] AND by ⊗, for every β ∈ D, β does not satisfy “(F(x) ∧ G(x))” on I₂ [⊗] [TC ∃]. By ⊗, either β does not satisfy “F(x)” on I₂ OR β satisfies “G(x)” on I₂ [TC →]. So, either β /∈ “F” on I₂ [⊕] [TC Atomic] OR β ∈ “G” on I₂ [⊕] [TC Atomic]. AND by ⊗, either β does not satisfy “F(x)” on I₂ OR β does not satisfy “G(x)” on I₂ [TC ∧]. So, either β /∈ “F” on I₂ [⊕] [TC Atomic] OR β /∈ “G” on I₂ [⊕] [TC Atomic]. Since 1 and 2 are the same condition and together are sufficient, let I₂ be as follows: D = {1}, “F”: {}, “G”: {}. Since I₁ makes the sentence true and I₂ makes the sentence false, this sentence is contingent.

2. Show “(F(a) ∧ ∀x(F(x) → G(x)) ∧ ¬G(a))” is L-false.

Proof: Suppose not; i.e., some I makes it true.
So, I makes “(F(a) ∧ ∀x(F(x) → G(x)))” true [⊕] AND I makes “¬G(a)” true [TC ∧]. Thus, I makes “G(a)” false [TC ¬][⊕].
By ⊕: I makes “F(a)” true [⊕] AND I makes “∀x(F(x) → G(x))” true [TC ∧][⊕].
By ⊕: on I there is some α ∈ D such that “a” designates α and α /∈ “G” [TC Atomic][⊕].
By ⊕ and ⊕: α ∈ “F” [TC Atomic][⊕].
By ⊕: on I, for every β ∈ D, β satisfies “(F(x) → G(x))” [TC ∀][⊕].
Since β is unrestricted, let β be α. Then, by ⊕ and substitution (α for β), α satisfies “(F(x) → G(x))” on I[⊕].
By ⊕: either α does not satisfy “F(x)” on I OR α satisfies “G(x)” on I[TC →].
⇒ Either α /∈ “F” OR α ∈ “G” [TC Atomic][⊕].
By ⊕ and ⊕: α /∈ “F”. [By disjunctive syllogism].
But, this contradicts ⊕. So, I is impossible. So, no I makes #2 true.
So, the sentence is L-false.

3. Show “((F(a) ∧ (a = b)) ∧ (b = c)) → F(c)” is L-true.

Proof: Suppose not; i.e., suppose some I makes it false.
So, I makes “((F(a) ∧ (a = b)) ∧ (b = c))” true [⊕] AND I makes “F(c)” false [⊗][TC →].
By \(\Box\): I makes “\(F(a)\)” true \([\Box]\) AND “\((a = b)\)” true \([\Box]\) AND “\((b = c)\)” true \([\Box][\text{TC } \land]\).
By \(\mathcal{O}\): on I there is some \(\alpha \in D\) such that “\(c\)” designates \(\alpha\) and \(\alpha \not\in F\) \([\text{TC Atomic}].\)
By \(\mathcal{O}\): “\(b\)” designates \(\alpha\) on I \([\text{TC } =][\Box].\)
By \(\mathcal{O}\) and \(\mathcal{O}\): “\(a\)” designates \(\alpha\) on I \([\text{TC } =][\Box].\)
By \(\mathcal{O}\) and \(\mathcal{O}\): \(\alpha \in “F”\) on I \([\text{TC Atomic}][\Box].\)
But, by \(\mathcal{O}\) and \(\mathcal{O}\), now we have \(\alpha \in “F”\) AND \(\alpha \not\in “F”\). That's a contradiction.
So, I is impossible.
So, no I makes the sentence false.
So, the sentence is L-true.

4. For homework!

5. Show “\(((a = b) \land (b = c)) \land (c = d)) \rightarrow (a = d)\)” is L-true.

Proof: Suppose not; i.e., suppose there is an I that makes it false.
Then I makes “\(((a = b) \land (b = c)) \land (c = d))\)” true \([\Box]\) AND I makes “\((a = d)\)” false \([\Box][\text{TC } \rightarrow].\)
By \(\Box\): I makes “\((a = b)\)” true AND “\((b = c)\)” true AND “\((c = d)\)” true \([\Box][\text{TC } \land].\)
By \(\Box\): on I there is some \(\alpha \in D\) such that “\(a\)”, “\(b\)”, “\(c\)”, and “\(d\)” all designate \(\alpha\) \([\Box][\text{TC } =].\)
But, by the Definition of Identity, on I for every \(\gamma \in D, <\gamma, \alpha> \in “=”.\)
This is a contradiction.
So, I is impossible.
So, no I makes the sentence false.
So, the sentence is L-true.

6. Show “\(\forall x(H(x,a) \rightarrow K(a,x)) \rightarrow (H(b,a) \rightarrow K(a,b))\)” is L-true.

Proof: Suppose not; i.e., suppose some I makes it false.
So \(\Box\) I makes “\(\forall x(H(x,a) \rightarrow K(a,x))\)” true AND \(\Box\) I makes “\((H(b,a) \rightarrow K(a,b))\)” false \([\text{TC } \rightarrow].\)
By \(\Box\): \(\Box\) I makes “\((H(b,a) \rightarrow K(a,b))\)” true AND \(\Box\) I makes “\((H(b,a) \rightarrow K(a,b))\)” true \([\Box][\text{TC } \land].\)
By \(\Box\): \(\Box\) on I for some \(\beta \in D\) and some \(\alpha \in D\) where “\(b\)” designates \(\beta\) and “\(a\)” designates \(\alpha\)
\(<\beta,\alpha> \in “H”\) \([\text{TC Atomic}].\)
By \(\Box\) and designations from \(\Box\): \([\Box] <\alpha, \beta> \not\in “K”\) \([\text{TC Atomic}].\)
By \(\Box\) and designations from \(\Box\): for every \(\gamma \in D, \gamma\) satisfies “\((H(x,a) \rightarrow K(a,x))\)” on I \([\text{TC } \forall].\)
\(\Rightarrow\) either \(\gamma\) does not satisfy “\((H(x,a) \rightarrow K(a,x))\)” on I OR \(\gamma\) satisfies “\((K(a,x))\)” on I \([\text{TC } \rightarrow].\)
\(\Rightarrow <\gamma, \alpha> \not\in “H”\) on I OR \(<\alpha, \gamma> \in “K”\) on I \([\Box][\text{TC Atomic}].\)
\(\gamma\) is unrestricted, so let \(\gamma\) be \(\beta\). By \(\Box\) and substitution, either \(<\beta, \alpha> \not\in “H”\) on I OR \(<\alpha, \beta> \in “K”\) on I.
By \(\Box\) and \(\Box\), \(<\beta, \alpha> \not\in “H”\) on I \([\Box].\)
But \(\Box\) contradicts \(\Box\).
So, I is impossible.
So, no I makes the sentence false. So, the sentence is L-true.
7. Show: \( \forall x \exists y R(x,y) \) \( \not\equiv \) \( \exists y \forall x R(x,y) \)

Proof: there is entailment iff some I makes

[1] “\( \forall x \exists y R(x,y) \)” true

[2] “\( \exists y \forall x R(x,y) \)” false.

By [1], for every \( \alpha \in D \), \( \alpha \) satisfies “\( \exists y R(x,y) \)” on I [TC \( \forall \)]. By [2], if I designates \( \alpha \) by ‘\( \mathbf{m} \)’, then ‘\( \forall x R(x,\mathbf{m}) \)’ is true on I [Def. Sat.]. Thus, for each \( \alpha \in D \), there is some \( \beta \in D \), such that \( <\alpha, \beta> \in “R” \) on I [TC Atomic]. By [2], for every \( \alpha \in D \), \( \alpha \) does not satisfy “\( \forall x R(x,y) \)” on I [TC \( \exists \)].

So, if I designates \( \alpha \) by ‘\( \mathbf{m} \)’, then ‘\( \forall x R(x,\mathbf{m}) \)’ is false on I [Def. Sat.]. So, there is some \( \gamma \in D \), such that \( \gamma \) does not satisfy ‘\( R(x,\mathbf{m}) \)” on I [TC \( \forall \)]. So, if I designates \( \alpha \) by ‘\( \mathbf{m} \)’ and I designates \( \gamma \) by ‘\( \mathbf{k} \)’, then ‘\( R(\mathbf{k},\mathbf{m}) \)” is false on I. [Def. Sat.]. Thus, for every \( \alpha \in D \), there is some \( \gamma \in D \), such that \( <\gamma, \alpha> \not\in “R” \) on I [TC Atomic]. Note that by [2] and [3], it cannot be that \( \alpha \) is always \( \beta \) which is the same object as \( \gamma \) [since if it were, then, substituting \( \gamma \) for both \( \alpha \) and \( \beta \), \( <\gamma, \gamma> \) would be both \( \in “R” \) and \( \not\in “R” \) on I]. So, on I, D must have more than one object in it. Set D={1,2}. The ordered pairs possible from this D are \( <1,1>, <1,2>, <2,1>, <2,2> \). By [2], either ‘\( <1,1> \in “R” \)” on I or \( <1,2> \in “R” \)” on I AND either ‘\( <2,1> \in “R” \)” on I or \( <2,2> \in “R” \)” on I. By [2], either ‘\( <1,1> \not\in “R” \)” on I or \( <2,1> \not\in “R” \)” on I AND either ‘\( <1,2> \not\in “R” \)” on I or \( <2,2> \not\in “R” \)” on I. So let I be as follows: D={1,2}, “R”: \{<1,1>, <1,2>, <2,1>, <2,2>\}. This I makes the sentence in the set on the lhs true and makes the sentence on the rhs false, thus demonstrating entailment.

8. Show \( \{ F(a), G(a), (a = b) \} \models (F(b) \rightarrow G(b)) \)

Proof: Suppose not, so some I makes:

[1] “\( F(a) \)” true

[2] “\( G(a) \)” true

[3] “\( (a = b) \)” true

[4] “\( (F(b) \rightarrow G(b)) \)” false.

By [3]: on I there is some \( \alpha \in D \) such that “\( a \)” and “\( b \)” both designate \( \alpha \) [TC =]. By [1] and designation from [3]: \( \alpha \in “F” \)” on I [TC Atomic]. By [2] and designation from [3]: \( \alpha \in “G” \)” on I [TC Atomic]. By [4]: I makes “\( F(b) \)” true AND “\( G(b) \)” false [TC \( \rightarrow \)]. By [2] and designation from [3]: \( \alpha \in “F” \)” on I AND \( \alpha \not\in “G” \)” on I [TC Atomic]. But by [7] and [8], this means that \( \alpha \in “G” \)” on I AND \( \alpha \not\in “G” \)” on I. But that’s impossible. So, no entailment I exists.

So, the entailment holds.

9. For Homework!
10. Show \( \forall x(F(x) \rightarrow G(x)), \forall y(G(y) \rightarrow H(y)), F(a) \) \( \models (H(a) \lor H(b)) \)

Proof: Suppose not, so some I makes:
- \([1]\) \(\forall x(F(x) \rightarrow G(x))\) true
- \([2]\) \(\forall y(G(y) \rightarrow H(y))\) true
- \([3]\) “F(a)” true
- \([4]\) “(H(a) \lor H(b))” false.

By \([1]\): on I for every \( \gamma \in D \), \( \gamma \) satisfies “(F(x) \rightarrow G(x))” \([TC \forall]\).

\( \Rightarrow \) \( \gamma \) does not satisfy “F(x)” on I OR \( \gamma \) satisfies “G(x)” on I \([TC \rightarrow]\).

\( \Rightarrow \gamma \notin “F” \) on I OR \( \gamma \in “G” \) on I \([TC Atomic]\).

By \([3]\): on I there is some \( \alpha \in D \) such that “a” designates \( \alpha \) and \( \alpha \in “F” \) \([TC Atomic]\).

By \([4]\) and the designation of \([\alpha]\), \( \alpha \notin “F” \) on I \([by \ disjunctive \ syllogism]\). But this contradicts \([\alpha]\).

So I is impossible. So, no entailment I exists. So the entailment holds.

11. Show \( \forall x(F(x) \rightarrow G(x,x)), \forall x \forall y(G(x,x) \rightarrow \neg F(y)) \) is consistent \([fixed \ typos \ on \ the \ problem \ sheet]\).

Proof: the set is consistent iff some I makes:
- \([1]\) \(\forall x(F(x) \rightarrow G(x,x))\) true
- \([2]\) \(\forall x \forall y(G(x,x) \rightarrow \neg F(y))\) true.

By \([1]\), for every \( \alpha \in D \), \( \alpha \) satisfies “(F(x) \rightarrow G(x,x))” on I \([3][TC \forall]\).

By \([3]\), either \( \alpha \) does not satisfy “F(x)” on I OR \( \alpha \) satisfies “G(x,x)” on I \([3][TC \rightarrow]\).

By \([2]\), either \( \alpha \notin “F” \) on I OR \( \alpha \subseteq “G” \) on I \([3][TC Atomic]\).

By \([2]\) and \([3]\), \( \alpha \notin “F” \) on I \([by \ disjunctive \ syllogism]\). But this contradicts \([\alpha]\).

So I is impossible. So, no entailment I exists. So the entailment holds.

12. Show \( \forall x \exists y R(x,y), \exists y \forall x R(x,y) \) is consistent.

Proof: the set is consistent iff some I makes:
- \([1]\) \(\forall x \exists y R(x,y)\) true
- \([2]\) \(\exists y \forall x R(x,y)\) true.
By ①, for every \( \alpha \in D \), \( \alpha \) satisfies “\( \exists y R(x,y) \)” on I [TC \( \forall \)]. So, if I designates \( \alpha \) by ‘\( m \)’, then ‘\( \exists y R(m,y) \)” is true on I [③][Def Sat.]. By ①, if I designates \( \alpha \) by ‘\( m \)’, then there is some \( \beta \in D \), such that \( \beta \) satisfies ‘\( R(m,y) \)” on I. So, if I designates \( \alpha \) by ‘\( m \)’ and I designates \( \beta \) by ‘\( n \)’, then ‘\( R(m,n) \)” is true on I. [Def. Sat.]. Thus, for each \( \alpha \in D \), there is some \( \beta \in D \), such that \( <\alpha, \beta> \in “R” \)” on I [③][TC Atomic].

By ②, for some \( \gamma \in D \), \( \gamma \) satisfies “\( \forall x R(x,y) \)” on I [TC \( \exists \)]. So, if I designates \( \gamma \) by ‘\( m \)’, then ‘\( \forall x R(x,m) \)” is true on I [Def. Sat.]. Thus, if I designates \( \gamma \) by ‘\( m \)’, then for all \( \alpha \in D \), \( \alpha \) satisfies ‘\( R(x,m) \)” on I [TC \( \forall \)]. So, there is some \( \gamma \in D \), such that for all \( \alpha \in D \), \( <\alpha, \gamma> \in “R” \)” on I [TC Atomic]. So, let I have \( D=\{1\}, \) “\( R \)”: \{<1,1>\}. This I makes both sentences true, so the set is consistent.

13. Show \( \{ R(a,b), R(b,a), \forall x \forall y (R(x,y) \rightarrow (R(x,x) \land R(y,y)), \neg R(a,a) \} \) is inconsistent.

Proof: Suppose not; some I makes:

[①] “\( R(a,b) \)” true
[②] “\( R(b,a) \)” true
[③] “\( \forall x \forall y (R(x,y) \rightarrow (R(x,x) \land R(y,y)) \)” true
[④] “\( \neg R(a,a) \)” true.

By ④, I makes “\( R(a,a) \)” false [⑤][TC \( \neg \)]

By ① and ②, on I there is some \( \alpha \in D \) and some \( \beta \in D \) such that

“\( a \)” designates \( \alpha \) AND
“\( b \)” designates \( \beta \) AND
\( <\alpha, \beta> \in “R” \) [⑥] AND
\( <\beta, \alpha> \in “R” \) [⑦][TC Atomic].

By ⑤, \( <\alpha, \alpha> \notin “R” \) [⑧][TC Atomic]

By ③, for every \( \gamma \in D \), \( \gamma \) satisfies “\( \forall y (R(x,y) \rightarrow (R(x,x) \land R(y,y))) \)” on I [TC \( \forall \)]. So, if I designates \( \gamma \) by ‘\( m \)’, then ‘\( \forall y (R(m,y) \rightarrow (R(m,m) \land R(y,y))) \)” is true on I [Def. Sat.]. Thus, if I designates \( \gamma \) by ‘\( m \)’, then for every \( \chi \in D \), \( \chi \) satisfies ‘\( R(m,y) \rightarrow (R(m,m) \land R(y,y))) \)” on I [⑨][TC \( \forall \)].

\( \Rightarrow \) on I, either \( \chi \) does not satisfy ‘\( R(m,y) \)” OR \( \chi \) satisfies ‘\( R(m,m) \land R(y,y))) \)” [TC \( \rightarrow \)].

\( \Rightarrow \) either \( <\gamma, \chi> \notin “R” \)” on I OR \( <\gamma, \gamma> \in “R” \)” on I and \( <\chi, \chi> \in “R” \)” on I [⑩][TC Atomic].

Note that \( \gamma \) and \( \chi \) are unrestricted in D, so let \( \gamma \) be \( \alpha \) and \( \chi \) be \( \beta \).

By ⑦ and substitution then,

Either \( <\alpha, \beta> \notin “R” \)” on I OR \( <\alpha, \alpha> \in “R” \)” on I AND \( <\beta, \beta> \in “R” \)” on I.

BUT \( <\alpha, \beta> \notin “R” \)” contradicts ⑧

AND \( <\alpha, \alpha> \in “R” \)” contradicts ⑧.

So, I is impossible.

So, the sent is inconsistent.

14. For Homework!
15. Show “¬∀x∃y¬G(x,y)” is semantically inequivalent to “∀x∃yG(x,y)” 

Proof: they are inequivalent iff either  
   [①] some I makes “¬∀x∃y¬G(x,y)” true AND makes “∀x∃yG(x,y)” false.  
   OR [②] some I makes “¬∀x∃y¬G(x,y)” false AND makes “∀x∃yG(x,y)” true. 

By ①, I1 makes “∀x∃y¬G(x,y)” false [③][TC ¬] AND for some α ∈ D, α does not satisfy “∃yG(x,y)” on I1 [⑤][TC ∀]. By ⑤, for some β ∈ D, β does not satisfy “∃y¬G(x,y)” on I [⑤][TC ∀]. By ⑤, if I1 designates α by ‘m’, then “∃yG(m,y)” is false on I1 [⑤][Def Sat.]. By ⑤, for every γ ∈ D, γ does not satisfy “G(m,y)” on I1 [⑤][TC ∃]. So, by ⑤, there is some α ∈ D such that, for every γ ∈ D, <α,γ> ∉ “G” on I1 [⑤][TC Atomic]. By ⑤, if I1 designates β by ‘n’, then “∃y¬G(n,y)” is false on I [Def. Sat.]. So, for every γ ∈ D, γ does not satisfy “¬G(n,y)” on I1 [TC 3]. Thus, γ satisfies “G(n,y)” on I1 [TC ¬]. Hence, there is some β ∈ D such that, for every γ ∈ D, <β,γ> “G” on I1 [⑤][TC Atomic]. By ⑤ and ⑤, α and β are distinct objects. Since γ is unrestricted, let γ be α or β. Then, by ⑤ and substitution, <α,β> ∉ “G” and <α,β> ∉ “G”. By ⑤ and substitution, <β,α> “G” and <β,α> member “G”. So let I1 have D={1,2}, “G”: {<2,1>, <2,2>}. This makes the first sentence true and the second sentence false, so they are semantically inequivalent. No need to consider case ②.

16. Show “∀x(F(x) → G(x))” is semantically equivalent to “∀y(¬G(y) → ¬F(y))”

Proof: Suppose not. Then there are two possible cases.
Case 1. Some I makes “∀x(F(x) → G(x))” true and “∀y(¬G(y) → ¬F(y))” false  
   ⇒ every α ∈ D satisfies “(F(x) → G(x))” on I [③][TC ∀]  
   and some γ ∈ D does not satisfy “(¬G(y) → ¬F(y))” on I [⑤][TC ∀].

By ③, either α does not satisfy “F(x)” on I [⑤] OR α satisfies “G(x)” on I [⑤][TC →]  
   AND by ⑤, γ satisfies “¬G(y)” on I [⑤] and does not satisfy “¬F(y)” on I [⑤][TC →].

By ⑤, γ does not satisfy “G(y)” on I [⑤] and by ⑤, γ satisfies “F(y)” on I [⑤][TC ¬].
By ⑤ and ⑤, α ∉ “F” on I or α ∉ “G” on I [TC Atomic] AND by ⑤ and ⑤, γ ∉ “G” on I and γ ∉ “F” on I [TC Atomic].

Since α is unrestricted, let α be γ.
By Substitution: on I, γ ∉ “F” or γ ∉ “G” AND γ ∉ “G” and γ ∉ “F”.
But these contradict. So, this case fails.

Case 2. Some I makes “∀x(F(x) → G(x))” false AND “∀y(¬G(y) → ¬F(y))” true.  
   ⇒ some α ∈ D does not satisfy “(F(x) → G(x))” on I [TC ∀]  
   AND every γ ∈ D satisfies “(¬G(y) → ¬F(y))” on I [TC ∀].

⇒ α satisfies “F(x)” on I and α does not satisfy “G(x)” on I [TC →]  
   AND either γ does not satisfy “¬G(y)” on I or γ satisfies “¬F(y)” on I [TC →].

⇒ γ satisfies “G(y)” on I or γ does not satisfy “F(y)” on I [TC ¬].
⇒ α ∈ “F” on I and α ∉ “G” on I AND either γ ∈ “G” on I or γ ∉ “F” on I [TC Atomic].
Since γ is unrestricted, let γ be α.
So, by substitution: on I, α ∈ “F” and α ∉ “G” AND either α ∈ “G” or α ∉ “F”.
But, these contradict. So this case fails.

Since both cases fail, there are no interpretations that give these sentences different truth-values.
So, these sentences are semantically equivalent.
17. Show “∀x(a = x)” is semantically equivalent to “∀z(z = a)”

Proof: Suppose not. Then, either there is some I₁ that makes
[①] “∀x(a = x)” true AND [②] “∀z(z = a)” false.
OR there is some I₂ that makes
[③] “∀x(a = x)” false AND [④] “∀z(z = a)” true.
Case 1: I₁ makes “∀x(a = x)” true and “∀z(z = a)” false.
By ①, for every α ∈ D, α satisfies “(a = x)” on I₁ [TC ∀]. So, if I₁ designates α by ‘m’,
then ‘(a = m)’ is true on I₁ [Def. Sat.]. So, if I₁ designates α by ‘m’, then I₁ designates α by “a” [⑤][TC =].
By ②, there is some β ∈ D, such that β does not satisfy “(z = a)” on I₁. So, if I₁
designates β by ‘m’, then ‘(m = a)’ is false on I₁ [Def. Sat.]. So, if I₁ designates β by ‘m’, then I₁ does not designate β by “a” [⑥][TC =].
⇒ By ⑤ and ⑥, α and β are distinct [TC =]. So, <β, α> ∉ “=” [⑧][Def. of Identity].
Since α is unrestricted, let α be β.
So, by ⑥ and substitution: <α, α> ∉ “=”. But that’s impossible. [Def. of Identity]
So, this case fails.
Case 2: I₂ makes “∀x(a = x)” false and “∀z(z = a)” true.
By ③, there is some α ∈ D, such that α does not satisfy “(a = x)” on I₂ [TC ∀]. So, if
I₂ designates α by ‘m’, then ‘(a = m)’ is false on I₂ [Def. Sat.]. So, if I₂ designates α by ‘m’, then I₂ does not designate α by “a” [⑧][TC =].
By ④, for every β ∈ D, β satisfies “(z = a)” on I₂ [TC ∀]. So, if I₂ designates β by ‘m’, then
‘(m = a)’ is true on I₂ [Def. Sat.]. So, if I₂ designates β by ‘m’, then I₂ designates β by “a” [⑧][TC =]. By ⑥ and ⑧, α and β are distinct [TC =]. So, <α, β> ∉ “=” [⑩][Def. of Identity].
Since β is unrestricted, let β be α.
So, by ⑧ and substitution, <α, α> ∉ “=”. But that’s impossible. [Def. of Identity]
So, this case fails.
Since both cases fail, no interpretation gives these sentences different truth-values.
So, these sentences are semantically equivalent.