Here I will go through another problem involving constructing an extensional interpretation to establish a logical/semantic fact, involving sentences with multiple quantifiers. Don’t worry that the explanation will go through lots of processing. This is just to make it clear what is going on. At the end I will give what would count as the kind of answer you should be able to construct.

Consider the task of showing that the following logical consequence claim is false.

\[
\{ \forall x \exists y R(x,y) \} \models \exists y \forall x R(x,y)
\]

The task is to show that this is a case of “enfailment”, rather than entailment. How do we do this? By constructing an interpretation, I, that makes all the sentences in the set on the left-hand side (lhs) of the “double turnstile” true and also makes the sentence on the right-hand side (rhs) of the “double turnstile” false. Since there is only one sentence in the set on the lhs, the interpretation, I, must make that sentence true while making the sentence on the rhs false.

So, the first thing we do is specify a domain, D. To try to keep things simple, let’s start, as usual, with having \( D = \{1\} \). If it turns out we need a bigger D, we’ll change to a different interpretation with additional items in the domain then. So, with this domain (and there being no atomic sentence letters to assign truth-values to and no names to assign objects from D to), we move to assigning an extension to each predicate. We have only one predicate, “\( R \)”, and it is a two-place predicate, so we will assign it a set of ordered-pairs as its extension. So, what should we put in the extension?

We want the lhs sentence, \( \forall x \exists y R(x,y) \), to come out true on I. It is a universal claim, so it is true provided every object in D satisfies the open (with respect to “\( x \)”) wff “\( \exists y R(x,y) \)” on I. Let’s assign the temporary name, “\( n \)”, to 1. Since 1 is the only object in D, every object in D satisfies “\( \exists y R(x,y) \)” on I provided the sentence (formed by replacing the open variable “\( x \)” with the name “\( n \)” “\( \exists y R(n,y) \)” is true on I. This is an existential sentence, so it is true on I provided some object from D satisfies the open (with respect to “\( y \)” wff “\( R(n,y) \)” on I. Since 1 is some object from D, if “\( R(n,n) \)” is true on 1, then “\( \exists y R(n,y) \)” will be true on 1. So, if we put \( <1,1> \) in the extension I assigns to “\( R \)” (in other words, if \( <1,1> \in \{R\} \)), this will make “\( R(n,n) \)” true on I, which will make the whole lhs sentence true on I.

However, if “\( R(n,n) \)” is true on I and 1 is the only object in D, then every object in D satisfies “\( R(x,n) \)” on I, which means that “\( \forall x R(x,n) \)” is true on I. And since “\( n \)” is still a temporary name for 1, then 1 satisfies “\( \forall x R(x,y) \)” on I. Since this means that some object from D satisfies “\( \forall x R(x,y) \)” on I, this means that “\( \exists y \forall x R(x,y) \)” is also true on I. But we wanted it to be false on I, in order to show enfailment. Does that mean we actually have entailment rather than enfailment? No, since enfailment just requires there be ONE interpretation out of the infinity of possible interpretations that makes the lhs sentence true and the rhs sentence false.

What our current situation shows is just that we don’t yet have an interpretation that shows this. (Notice that we do, however, have an interpretation that shows the two sentences are consistent. Can you see why?)

So, we need a different interpretation, I’, with a bigger domain. Let’s try \( D = \{1,2\} \). With this D, what does it take to make the lhs sentence true on \( I' \)? Recall that for “\( \forall x \exists y R(x,y) \)” to be true on \( I' \), every object in D must satisfy the open wff “\( \exists y R(x,y) \)” on \( I' \). If we still make \( <1,1> \in \)
“R”, then (with “n” still a temporary name for 1 on I) “R(n,n)” is true on I’, so 1 satisfies the open wff “R(n,y)” on I’ and thus “∃yR(n,y)” is true on I’. Furthermore, 1 also satisfies the open wff “∃yR(x,y)” on I’. But since 2 is now in D along with 1, we need 2 to satisfy this open wff on I’ as well. Let’s assign the temporary name “m” to 2 (and assign the temporary name “n” to 1). So, we need “∃yR(m,y)” to be true on I’ (along with “∃yR(n,y)” being true on I’). The sentence “∃yR(m,y)” is true on I’ provided some object from D satisfies the open wff “R(m,y)” on I’. What this means is that we need the extension I’ assigns to “R” to include an ordered-pair that starts with 2 (as in, <2,…>). Now, we could either put 1 in the second spot in the pair, or we could put 2 in there. Which should we do?

Well, recall that we want to make the sentence “∃y∀xR(x,y)” false on I’, which means every object in D must not satisfy the open wff “∀xR(x,y)” on I’. If we put 1 in the second spot in the ordered-pair that starts with 2, then both <1,1> ∈ “R” and <2,1> ∈ “R”, so both “R(n,n)” and “R(m,n)” would be true on I’. But that would mean that both 1 and 2 would satisfy the open wff “R(x,n)” on I’, and since they are everything the domain contains, that would make the sentence “∀xR(x,n)” true on I’. But if that sentence is true on I’, then since “n” is a temporary name for 1, 1 satisfies the open wff “∀xR(x,y)” on I’. Since some object from D would satisfy “∀xR(x,y)” on I’, the sentence “∃y∀xR(x,y)” would again be true on I’, rather than false on I’, as we want it to be.

So, we need an ordered-pair that starts with 2 in the extension of “R”, but it can’t be <2,1>. So, instead, let’s put in <2,2>. We already determined that this will make the full lhs sentence true (given that I’ already put <1,1> in the extension it assigns to “R”). How about the full rhs sentence? Since <2,2> ∈ “R” and <1,1> ∈ “R” and that’s everything in the extension of “R”, this means both “R(m,m)” and “R(n,n)” are true on I’ (but no other “atomic substitution instance” for “R(x,y)” will be). However, while 1 satisfies the open wff “R(x,n)” on I’, 2 does not satisfy that open wff on I’, so the sentence “∀xR(x,n)” is false on I’. And while 2 satisfies the open wff “R(x,m)” on I’, 1 does not satisfy that open wff on I’, which makes the sentence “∀xR(x,m)” false on I’. Since both of these sentences are false on I’, then, since “n” designates 1 on I’ and “m” designates 2 on I’, neither 1 nor 2 satisfy the open wff “∀xR(x,y)” on I’. (If one or the other did, then given our temporary name assignments, either “∀xR(x,n)” or “∀xR(x,m)” would be true on I’. But neither is.) So, we can see that no object from D satisfies the open wff “∀xR(x,y)” on I’. Thus, I’ makes “∃y∀xR(x,y)” false, as desired.

Ok, so that’s the reasoning. From this we see that there is at least one interpretation, I’, that succeeds in showing entailment here, where I’ is as follows.

D= {1, 2}
R: {<1,1>, <2,2>}

Here is an explanation of why this interpretation is a counter-example.

If we assign the temporary name “n” to 1 and assign the temporary name “m” to 2, then, since <1,1> ∈ “R”, the sentence “R(n,n)” is true on I’ [TC Atom]. So, 1 ∈ D satisfies “R(n,y)” on I’, and thus the sentence “∃yR(n,y)” is true on I’ [TC ∃]. Since “n” designates 1 on I’, this also means that 1 ∈ D satisfies “∃yR(x,y)” on I’. Since <2,2> ∈ “R”, the sentence “R(m,m)” is true on I’ [TC Atom]. So 2 ∈ D satisfies “R(m,y)” on I’, and so the sentence “∃yR(m,y)” is true on I’ [TC ∃]. Since “m” designates 2 on I’, this also means that 2 ∈ D satisfies “∃yR(x,y)” on I’. Since we now see that both 1 and 2 satisfy “∃yR(x,y)” on I’, and they are all that is in D, we now see
that every object in D satisfies “∃yR(x,y)” on I’, so the sentence “∀x∃yR(x,y)” is true on I’ [TC ∀]. That’s the lhs. Moving to the rhs, since <2,1> ∉ “R”, the sentence “R(m,n)” is false on I’ [TC Atom]. This means that 2 ∈ D does not satisfy “R(x,n)” on I’. So, “∀xR(x,n)” is false on I’ [TC ∀]. Since <1,2> ∉ “R”, the sentence “R(n,m)” is false on I’ [TC Atom]. This means that 1 ∈ D does not satisfy “R(x,m)” on I’, and so “∀xR(x,m)” is false on I’ [TC ∀]. Since “n” designates 1 on I’ and “m” designates 2 on I’, both 1 and 2 do not satisfy “∀xR(x,y)” on I’. Thus, I’ makes “∃y∀xR(x,y)” false [TC ∃], as desired. Since I’ makes every sentence in the set on the lhs true and I’ makes the sentence on the rhs false, I’ is a counter-example and shows this case to be an “enfailment”.