MI Examples Solutions
PHIL 422
Dove/Woodbridge

1. Prove that at any conference, the number of people who have shaken an odd number of hands (assuming all handshakes are normal) is even. This will be a Proof by Induction on the Number of Handshakes.

Preliminaries: Let the collection of conference attendees be $C$. Partition $C$ into two subsets, $E$ and $O$, such that $E$ is the collection of conference attendees who have shaken an even number of hands, and $O$ is the collection of conference attendees who have shaken an odd number of hands. Let $O^n$ give the partition of $C$ of people in $O$ at the level of $n$ handshakes. And let $|O^n|$ be the size of the extension of $O$ for a given number of handshakes, $n$, that have so far occurred at the conference. We will show that for any number of handshakes, $|O^n|$ is even, i.e. $= 2k$ for some $k$.

Proof:

**Basis**: When no one has shaken hands, $|C| = |E^0|$, and $|O^0| = 0$. Since zero is, by convention, even, this case is confirmed. [Or: $|O^0| = 2k$ (where $k = 0$).]

**Inductive Hypothesis**: Suppose that for every number of handshakes, $m$, where $m \leq n$, $|O^m| = 2k$.

We need to show that when the number of handshakes is $n + 1$, $|O^{n+1}| = 2k$ (for some $k$).

There are four cases. Let $i$ and $j$ designate the attendees who shake hands in the transition from there being $n$ handshakes to there being $n + 1$ handshakes.

Case 1: $i \in E$ and $j \in E$. Since both have shaken an even number of hands, then both exit $E$ and enter $O$, i.e., now $i \in O$ and $j \in O$. By IH, at level $n$, $|O^n| = 2k$. At level $n + 1$, $|O^{n+1}| = 2k + 1 + 1 = 2k + 2 = 2(k + 1)$, which is even. So, this case is confirmed.

Case 2: $i \in E$ and $j \in O$. When $i$ and $j$ shake hands, $i$ enters $O$ and $j$ exits $O$. Since, by IH, $|O^n| = 2k$, $|O^{n+1}| = 2k + 1 – 1 = 2k$, which is even. So, this case is confirmed.

Case 3: $i \in O$ and $j \in E$. When $i$ and $j$ shake hands $i$ exits $O$ and $j$ enters $O$. Since, by IH, $|O^n| = 2k$, $|O^{n+1}| = 2k + 1 – 1 = 2k$, which is even. So, this case is confirmed.

Case 4: $i \in O$ and $j \in O$. When $i$ and $j$ shake hands $i$ exits $O$ and $j$ exits $O$. Since, by IH, $|O^n| = 2k$, $|O^{n+1}| = 2k – 1 – 1 = 2k – 2 = 2(k – 1)$, which is even. So, this case is confirmed.

**Inductive Result**: So, if, for every number, $m \leq n$, of handshakes, $|O^m| = 2k$, then $|O^{n+1}| = 2k$ (for some $k$). [This is the ladder principle, LP.]

**Ladder Conclusion**: Since Basis shows that $|O^0| = 2k$, LP gets us $|O^1| = 2k$ (for some $k$). Since $|O^n| = 2k$ (for some $k$) when $n = 0$ and $n = 1$, LP yields $|O^2| = 2k$ (for some $k$). Since $|O^n| = 2k$ (for some $k$) when $n = 0$ and $n = 1$ and $n = 2$, LP yields $|O^3| = 2k$ (for some $k$). This “ladder” continues all the way up the natural numbers, to get us $|O^n| = 2k$ (for some $k$) for all $n$, which is what we wanted to show.
2. Show that for any wff of FOL, the number of atoms in the wff is the number of binary 
operators in the wff, plus one. (Show that for any ‘φ’ [i.e., for ‘φ’ of any complexity n], \(|φ|_a = |φ|_b + 1\).) This will be a Proof by Induction on the Number of Operators.

Preliminaries: Let \(|φ|_a = \# \) of atoms in \(φ\). Let \(|φ|_b = \# \) of binary operators in ‘φ’. Let \(|φ|_o = \) the number of operators (including both binary and unary operators) in ‘φ’. Let ‘@’ be any unary 
operator, i.e., ‘¬’ or ‘∃’ or ‘∀’. Let ‘∗’ be any binary operator, i.e., ‘∧’ or ‘∨’ or ‘⇒’ or ‘⇔’.

**Basis:** ‘φ’ is atomic, i.e., \(|φ|_o = 0\). Then \(|φ|_a = 1\) and \(|φ|_b = 0\). So, when \(|φ|_o = 0\), \(|φ|_a = |φ|_b + 1\).

**Inductive Hypothesis:** For \(|φ|_o \leq n\), let \(|φ|_a = |φ|_b + 1\). We need to show that for \(|φ|_o = n + 1\), \(|φ|_a = |φ|_b + 1\).

There are two cases. In one (kind of) case, the main operator of ‘φ’ is unary. In the other (kind of) case, the main operator of ‘φ’ is binary.

**Case 1:** \(|φ|_o = n + 1\) and ‘φ’ is ‘@ψ’ for some unary operator ‘@’ and wff ‘ψ’. We need to show that \(|@ψ|_a = |@ψ|_b + 1\). Since \(|@ψ|_o = n + 1\), \(|ψ|_o = n\). So, \(1\) \(|ψ|_a = |ψ|_b + 1\) (by IH). Moreover, since the main operator of ‘φ’ is ‘@’, \(2\) \(|ψ|_o = |@ψ|_o\) because unary operations do not add atoms. Also, \(3\) \(|ψ|_b = |@ψ|_b\) because unary operations do not add binary operations. Now we can replace the identities from \(2\) and \(3\) in statement \(1\). This gives: \(|@ψ|_a = |@ψ|_b + 1\), which is what we wanted to prove.

**Case 2:** \(|φ|_o = n + 1\), and ‘φ’ is ‘(ψ ∗ χ)’ for some binary operation ‘∗’ and wffs ‘ψ’ and ‘χ’. We want to show that \(|(ψ ∗ χ)|_a = |(ψ ∗ χ)|_b + 1\). Since \(|(ψ ∗ χ)|_o = n + 1\), both \(|ψ|_o ≤ n\) and \(|χ|_o ≤ n\). Hence, \(1\) \(|ψ|_a = |ψ|_b + 1\), and \(2\) \(|χ|_a = |χ|_b + 1\) (by hypothesis). Note well that \(3\) \(|ψ|_b = |ψ|_a + |χ|_a\) [because the operator combines the two wffs without adding its own wff] and \(4\) \(|ψ|_a + |χ|_a = |ψ|_b + |χ|_b + 1\) [because the operator adds all of the binary operation in ‘ψ’ to the binary operation in ‘χ’ PLUS the operation ‘∗’ itself]. Let’s start with \(|ψ|_a\) and add to it \(|χ|_a\), i.e., \(|ψ|_a + |χ|_a = ?\). Given \(1\) and \(2\), \(|ψ|_a + |χ|_a = |ψ|_b + 1 + |χ|_b + 1\) [call this \(5\)]. Rearrange the terms of \(5\): \(|ψ|_a + |χ|_a = |ψ|_b + |χ|_b + 1 + 1\) Now in the rearranged \(5\), replace \(|ψ|_a + |χ|_a\) with \(|(ψ ∗ χ)|_a\) because of \(3\) and replace \(|ψ|_b + |χ|_b + 1\) with \(|(ψ ∗ χ)|_b\) because of \(4\): \(|(ψ ∗ χ)|_a = |(ψ ∗ χ)|_b + 1\). And this is what we wanted to show.

**Inductive Conclusion:** Since both cases lead to \(|φ|_a = |φ|_b + 1\) for \(|φ|_o = n + 1\), we conclude: If \(|φ|_o = n\) and \(|φ|_a = |φ|_b + 1\), then, for \(|φ|_o = n + 1\), \(|φ|_a = |φ|_b + 1\). [This is the ladder principle, LP.]

**Ladder Conclusion:** Since \(Basis\) shows that for \(|φ|_o = 0\), \(|φ|_a = |φ|_b + 1\), then by LP, for \(|φ|_o = 1\), \(|φ|_a = |φ|_b + 1\). Since for \(|φ|_o = 0\) and \(|φ|_a = |φ|_b + 1\), then by LP, for \(|φ|_o = 2\), \(|φ|_a = |φ|_b + 1\). Since for \(|φ|_o = 0\) and \(|φ|_a = |φ|_b + 1\) and \(|φ|_o = 2\), \(|φ|_a = |φ|_b + 1\), then by LP, for \(|φ|_o = 3\), \(|φ|_a = |φ|_b + 1\). And so on, up through all the levels, establishing that for any n s.t. \(|φ|_o = n\), \(|φ|_a = |φ|_b + 1\), which is what we wanted to show!
3. Let Mr. T, \( \varepsilon^{T}_{\leq} \), be an interpretation that makes every atomic sentence of FOL true. Show that Mr. T makes any wff \( \phi \) true where \( \phi \) does NOT contain any unary operators.

Proof by Mathematical Induction on the Binary Complexity of \( \phi \).

**Basis**: \( |\phi|_{b} = 0 \). Then \( \phi \) is atomic. Since Mr. T makes all atomic sentences true, Mr. T makes \( \phi \) true.

**Inductive Hypothesis**: For \( |\phi|_{b} \leq n \), let Mr. T make \( \phi \) true. Show that for \( |\phi|_{b} = n + 1 \), Mr. T makes \( \phi \) true. This means that \( \phi \) is \( (\psi \land \chi) \) for all binary operators \( \ast \) and for wffs \( \psi \) and \( \chi \). There are four cases to consider: \( \ast \) is \( \land \), \( \ast \) is \( \lor \), \( \ast \) is \( \rightarrow \) and \( \ast \) is \( \leftrightarrow \).

Case 1: \( |\phi|_{b} = n + 1 \), and \( \phi \) is \( (\psi \land \chi) \). Since \( \land \) is binary, \( |\psi|_{b} \leq n \) and \( |\chi|_{b} \leq n \). So, Mr. T makes \( \psi \) true by hypothesis and Mr. T makes \( \chi \) true by hypothesis. Therefore, Mr. T makes \( (\psi \land \chi) \) true. This is what we wanted to show!

Case 2: \( |\phi|_{b} = n + 1 \), and \( \phi \) is \( (\psi \lor \chi) \). Since \( \lor \) is binary, \( |\psi|_{b} \leq n \) and \( |\chi|_{b} \leq n \). So, Mr. T makes \( \psi \) true by hypothesis and Mr. T makes \( \chi \) true by hypothesis. Therefore, Mr. T makes \( (\psi \lor \chi) \) true. This is what we wanted to show!

Case 3: \( |\phi|_{b} = n + 1 \), and \( \phi \) is \( (\psi \rightarrow \chi) \). Since \( \rightarrow \) is binary, \( |\psi|_{b} \leq n \) and \( |\chi|_{b} \leq n \). So, Mr. T makes \( \psi \) true by hypothesis and Mr. T makes \( \chi \) true by hypothesis. Therefore, Mr. T makes \( (\psi \rightarrow \chi) \) true. This is what we wanted to show!

Case 4: \( |\phi|_{b} = n + 1 \), and \( \phi \) is \( (\psi \leftrightarrow \chi) \). Since \( \leftrightarrow \) is binary, \( |\psi|_{b} \leq n \) and \( |\chi|_{b} \leq n \). So, Mr. T makes \( \psi \) true by hypothesis and Mr. T makes \( \chi \) true by hypothesis. Therefore, Mr. T makes \( (\psi \leftrightarrow \chi) \) true. This is what we wanted to show!

**Inductive Conclusion**: If \( |\phi|_{b} \leq n \) and Mr. T makes \( \phi \) true, then for \( |\phi|_{b} = n + 1 \), Mr. T makes \( \phi \) true. [This is the ladder principle, LP.]

**Ladder Conclusion**: Since Basis show that for \( |\phi|_{b} = 0 \), Mr. T makes \( \phi \) true, then by LP, for \( |\phi|_{b} = 1 \), Mr. T makes \( \phi \) true. Since \( |\phi|_{b} = 0 \) and \( |\phi|_{b} = 1 \), Mr. T makes \( \phi \) true, then by LP, for \( |\phi|_{b} = 2 \), Mr. T makes \( \phi \) true. Since for \( |\phi|_{b} = 0 \) and \( |\phi|_{b} = 1 \) and \( |\phi|_{b} = 2 \), Mr. T makes \( \phi \) true, then by LP, for \( |\phi|_{b} = 3 \), Mr. T makes \( \phi \) true. And so on, up through all the levels, meaning that for any \( n \) where \( |\phi|_{b} = n \), Mr. T makes \( \phi \) true!