Definitions and Concepts

$\Sigma$ is an arbitrary set of sentences of FOL

$T[\Sigma] = \Sigma$’s completed tree

$T_n[\Sigma] =$ an nth-level tree (not necessarily complete) for $\Sigma$.

$B$ is a branch on $T_n[\Sigma]$.

$B$ is the set of sentences on the branch, $B$.

$|B|$ is the length of $B$. The length is measured by the number of “rule-application” lines on $B$.

An nth-level tree, $T_n[\Sigma]$, is a tree whose longest branch has length $|B| = n$.

Explanation

We have a system (a deductive apparatus) for testing sets of sentences of FOL for a variety of syntactic properties. We don’t know whether the method is semantically reliable. We have, in fact, pretended that the system is semantically reliable, but we don’t yet know it. There are two things we’d like to know about the semantic reliability of our system.

We want to know that, for any set $\Sigma$, when $\Sigma$ is consistent, then $T[\Sigma]$ is open; AND, when $\Sigma$ is inconsistent, then $T[\Sigma]$ closes. We’ll break these into two separate results.

Downward Adequacy Theorem (DAT): For any arbitrary set of sentences, $\Sigma$, if $\Sigma$ is consistent, then the completed tree of $\Sigma$ will contain at least one open branch.

Roughly, this means that if the top of a tree is consistent, then the repeated application of the tree rules will not result in a contradiction. In other words, contradictions can’t sneak into a tree.

Upward Adequacy Theorem (UAT): For any set of sentences, $\Sigma$, if $\Sigma$ is inconsistent, then every completed tree of $\Sigma$ will contain no open branches.

This means that inconsistencies will always be discovered. If the top of a tree is inconsistent, then the repeated application of our rules will result in an explicit contradiction on every branch.

It turns out that, stated in this way, UAT is difficult to prove. But, there is at least one equivalent formulation of this claim that is easier to prove, the contrapositive of UAT.

UAT*: For any set of sentences, $\Sigma$, if some completed tree of $\Sigma$ is open, then $\Sigma$ is consistent.

If these two claims, DAT and UAT are true of a system, then the system will determine for any set of sentences of FOL whether the set is consistent (or not). This is equivalent to the system being Sound (DAT) and Complete (UAT).
**Proof of DAT for FOL**

We want to show that for any set of sentence, \( \Sigma \), if \( \Sigma \) is consistent, then \( T[\Sigma] \) is open.

We will give prove this conditional via conditional proof. So, we will assume that \( \Sigma \) is consistent, i.e., that there is an interpretation, \( I \), that makes all of \( \Sigma \) true. We will then show that on this assumption it follows that \( T[\Sigma] \) is open, i.e., that this tree has at least one open branch, \( B \).

Proof by Mathematical Induction

**Induction Set:** The set of all the trees (complete or incomplete) that can be generated, starting with \( \Sigma \) (i.e., all \( T_0[\Sigma] \)).

**Partition:** The set of \( T_n[\Sigma] \) is divided into levels in terms of the length of each tree’s longest branch. (So, level \( k \) contains all the \( T_k[\Sigma] \).)

**Basis:** \( |B|=0 \), i.e., no “rule-application” lines. This means that \( B_s=\Sigma \). We want to show that \( T_0[\Sigma] \) is open and that every sentence on this open branch (i.e., every member of \( \Sigma \)) is true on some interpretation, \( I \).

That every sentence of \( B_s \) is true on some \( I \) is a direct consequence of the assumption that \( \Sigma \) is consistent. Moreover, because \( \Sigma \) is consistent, it follows that \( T_0[\Sigma] \) is open. The only way the tree could close is if, for some atomic sentence, ‘\( \phi \)’, \( \Sigma \) contained both ‘\( \phi \)’ and ‘\( \neg \phi \)’. But then if one were true on \( I \), the other would be false on \( I [TC \neg] \), and \( \Sigma \) would not be consistent. So \( T_0[\Sigma] \) must be open. And this is what we wanted to show.

**Induction Step:**

Inductive Hypothesis (IH): Suppose that \( T_n[\Sigma] \) is open at all lengths \( |B| \leq k \) (i.e., where \( n \leq k \)) and that there is some interpretation, \( I \), that makes all of the sentences in the set \( B_s \), the sentences on \( B \) from every level \( n \) up to and including level \( k \), true.

We want to show that, assuming IH, \( T_{k+1}[\Sigma] \) is open and that the resulting sentence(s), added to \( B_s \) from level \( k+1 \), is/are true on \( I \). This involves considering all of the ways a branch can go from \( |B|=k \) to \( |B|=k+1 \). There are 15 (really, 14) ways, one for each rule from our deductive apparatus.

Case 1: \( \neg \neg \phi \) on \( B \) at level \( n \leq k \) and we apply the double negation rule to get to \( k+1 \)

\[
\neg \phi \text{ on } B \text{ at level } k+1 \quad \text{We want to show both that } \phi \text{ is true on } I \text{ and that } T_{k+1}[\Sigma] \text{ is open at level } k+1.
\]

Since ‘\( \neg \phi \)’ is on \( B \) at level \( n \leq k \), then it is true on \( I \), by hypothesis (IH). Thus, ‘\( \neg \phi \)’ is false on \( I [TC \neg] \). Hence, ‘\( \phi \)’ is true on \( I [TC \neg] \). This is the first thing we wanted to show.

To show that \( T_{k+1}[\Sigma] \) is open, consider what it would take for \( B \) to close at level \( k+1 \). \( B \) would close iff ‘\( \neg \phi \in B_s \). But, since, by hypothesis, ‘\( \neg \phi \)’ is true on \( I \), making ‘\( \neg \phi \)’ false on \( I [TC \neg] \), it follows that ‘\( \neg \phi \in B_s \), since (by IH) \( I \) makes all of the sentences of \( B_s \) true. Thus, \( T_{k+1}[\Sigma] \) is open.
Case 2: \((\phi \land \psi)\) on B at level \(n \leq k\), and we apply the conjunction rule to get to \(k + 1\)

\[
\begin{align*}
\phi & \quad \text{on B at level } k + 1 \text{ AND} \\
\psi & \quad \text{on B at level } k + 1
\end{align*}
\]

We want to show both that ‘\(\phi\)’ and ‘\(\psi\)’ are true on I and that \(T_{k+1}[\Sigma]\) is open.

Since ‘\((\phi \land \psi)\)’ is on B at level \(n \leq k\), it is true on I by IH. But that means that ‘\(\phi\)’ is true on I and ‘\(\psi\)’ is true on I [TC \(\land\)], which is what we wanted to show first.

To show that \(T_{k+1}[\Sigma]\) is open, consider what it would take for B to close at level \(k + 1\). B would close iff either \(\neg \phi \in B_s\) or \(\neg \psi \in B_s\) at some level \(n \leq k\). Since we assumed that ‘\((\phi \land \psi)\)’ is true on I, and that I makes both ‘\(\phi\)’ and ‘\(\psi\)’ true on I, both ‘\(\neg \phi\)’ and ‘\(\neg \psi\)’ are false on I [TC \(\neg\)]. So, since I makes everything in \(B_s\), every sentence on B up to and including the sentence at level \(k\), it follows that neither ‘\(\neg \phi\)’ nor ‘\(\neg \psi\)’ is on B. But that means that the branch \(T_{k+1}[\Sigma]\) remains open.

Case 3: \((\phi \lor \psi)\) on B at level \(n \leq k\), and we apply the disjunction rule to get to \(k + 1\)

\[
\Delta \quad \text{on B at level } k + 1
\]

[While the rule for “\(\lor\)” branches, rather than stems, we are looking at only one branch or the other to show openness. So \(\Delta\) is either the left branch, \(\phi\), or right branch, \(\psi\), depending on why the starting disjunction is true on I.]

We want to show that the wff at level \(k + 1\) on B is true on I, i.e., we want to show that some \(\Delta\) on B is true on I at level \(k + 1\) AND that \(T_{k+1}[\Sigma]\) is open. Subcases to consider are from [TC \(\lor\)].

Subcase A: ‘\((\phi \lor \psi)\)’ is true on I because both ‘\(\phi\)’ is true on I and ‘\(\psi\)’ is true on I. Since, both are true on I, let \(\Delta\) be either one. So, by assumption, \(\Delta\) is true on I. Since ‘\((\phi \lor \psi)\)’, ‘\(\phi\)’, and ‘\(\psi\)’ are all true on I, it follows that \(\neg \phi \notin B_s\) and \(\neg \psi \notin B_s\). Hence, \(T_{k+1}[\Sigma]\) is open because either ‘\(\neg \phi\)’ or ‘\(\neg \psi\)’ would have to be on B (\(\in B_s\)) for B to close.

Subcase B: ‘\((\phi \lor \psi)\)’ is true on I because ‘\(\phi\)’ is true on I and ‘\(\psi\)’ is false on I. Since ‘\(\phi\)’ is true on I and ‘\(\psi\)’ is false on I, let \(\Delta\) be \(\phi\). So, \(\Delta\) is true on I. Since ‘\((\phi \lor \psi)\)’ and ‘\(\phi\)’ are true on I, and ‘\(\psi\)’ is false on I, it follows that \(\neg \phi \notin B_s\) (otherwise ‘\((\phi \lor \psi)\)’ would be false on I [TC \(\lor\)]). Hence, \(T_{k+1}[\Sigma]\) is open because only ‘\(\neg \phi\)’ being on B (\(\in B_s\)) would close B.

Subcase C: ‘\((\phi \lor \psi)\)’ is true on I because ‘\(\psi\)’ is true on I and ‘\(\phi\)’ is false on I. Since, ‘\(\psi\)’ is true on I and ‘\(\phi\)’ is false on I, let \(\Delta\) be ‘\(\psi\)’. So, \(\Delta\) is true on I. Since ‘\((\phi \lor \psi)\)’ and ‘\(\psi\)’ are true on I and ‘\(\phi\)’ is false on I, it follows that \(\neg \psi \notin B_s\) (otherwise ‘\((\phi \lor \psi)\)’ would be false on I [TC \(\lor\)]). Hence, \(T_{k+1}[\Sigma]\) is open because only ‘\(\neg \psi\)’ being on B (\(\in B_s\)) would close B.

Since this exhausts the subcases, it follows that \(\Delta\) is true on I for at least one branch extending from the end of an open branch of length \(|B| = k\) (in virtue of our processing a disjunction occurring on B at some level \(n \leq k\)), and that \(T_{k+1}[\Sigma]\) remains open.
Case 4: \((\phi \rightarrow \psi)\) on level \(n \leq k\), and we apply the conditional rule to get to level \(k + 1\)

\[
\triangle \text{ on level } k + 1 \quad [\triangle \text{ will be either the left branch, } \neg \phi, \text{ or the right branch, } \psi, \\
\text{ depending upon why the conditional is true on I.}]
\]

[We’ll forego explanation for the rest of the sentential cases, and simply do the rest of the proof.]

Subcase A: ‘\((\phi \rightarrow \psi)\)’ is true on I because both ‘\(\phi\)’ is true on I and ‘\(\psi\)’ is true on I. So ‘\(\neg \phi\)’ is false on I \([TC \neg]\). Since ‘\(\neg \phi\)’ is false on I and ‘\(\psi\)’ is true on I, let \(\triangle\) be ‘\(\psi\)’. Then, by assumption, \(\triangle\) is true on I. Since ‘\((\phi \rightarrow \psi)\)’, ‘\(\phi\)’, and ‘\(\psi\)’ are true on I, it follows that ‘\(\neg \psi \notin B_s\). Hence, \(T_{k+1}[\Sigma]\) is open, since only ‘\(\neg \psi\)’ being on B \((\in B_s)\) could close B.

Subcase B: ‘\((\phi \rightarrow \psi)\)’ is true on I because I makes ‘\(\phi\)’ false and ‘\(\psi\)’ true. So, ‘\(\neg \phi\)’ is true on I \([TC \neg]\). Since ‘\(\neg \phi\)’ and ‘\(\psi\)’ are both true on I, let \(\triangle\) be either one. Then \(\triangle\) is true on I. Since ‘\((\phi \rightarrow \psi)\)’, ‘\(\neg \phi\)’, and ‘\(\psi\)’ are true on I, it follows that \(\phi \notin B_s\) and ‘\(\neg \psi \notin B_s\) (since I makes everything in B true). Hence, \(T_{k+1}[\Sigma]\) is open since only ‘\(\phi\)’ or ‘\(\neg \psi\)’ being on B \((\in B_s)\) would close B.

Subcase C: ‘\((\phi \rightarrow \psi)\)’ is true on I because I makes ‘\(\phi\)’ false and makes ‘\(\psi\)’ false. So ‘\(\neg \phi\)’ is true on I \([TC \neg]\). Since ‘\(\psi\)’ is false on I and ‘\(\neg \phi\)’ is true on I, let \(\triangle\) be ‘\(\neg \phi\)’. So, by assumption, \(\triangle\) is true on I. Since ‘\((\phi \rightarrow \psi)\)’ and ‘\(\neg \phi\)’ are true on I and ‘\(\psi\)’ is false on I, it follows that ‘\(\phi \notin B_s\) (otherwise ‘\((\phi \rightarrow \psi)\)’ would be false on I). Hence, \(T_{k+1}[\Sigma]\) is open because only ‘\(\phi\)’ being on B \((\in B_s)\) would close B.

Since this exhausts the subcases, it follows that some \(\triangle\) is true on I and \(T_{k+1}[\Sigma]\) is open.

Case 5: \((\phi \leftrightarrow \psi)\) on level \(n \leq k\) and we apply the biconditional rule to get to level \(k + 1\)

\[
\square \text{ on level } k + 1 \quad \text{AND} \\
\triangle \text{ on level } k + 1 \quad [\text{what } \square \text{ and } \triangle \text{ are depends on which branch we go down,} \\
\text{which depends on why the biconditional is true on I}]
\]

Subcase A: ‘\((\phi \leftrightarrow \psi)\)’ is true on I because I makes both ‘\(\phi\)’ and ‘\(\psi\)’ true. Since, ‘\(\phi\)’ is true on I and ‘\(\psi\)’ is true on I, let \(\square\) be ‘\(\phi\)’ and \(\triangle\) be ‘\(\psi\)’. So, \(\square\) and \(\triangle\) are both true on I. Since ‘\((\phi \leftrightarrow \psi)\)’, ‘\(\phi\)’, and ‘\(\psi\)’ are true on I, it follows that ‘\(\neg \phi \notin B_s\) and ‘\(\neg \psi \notin B_s\) (otherwise, ‘\((\phi \leftrightarrow \psi)\)’ would be false on I). Hence, \(T_{k+1}[\Sigma]\) is open, since only ‘\(\neg \phi\)’ or ‘\(\neg \psi\)’ being on B \((\in B_s)\) would close B.

Subcase B: ‘\((\phi \leftrightarrow \psi)\)’ is true on I because I makes both ‘\(\phi\)’ and ‘\(\psi\)’ false. So, I makes ‘\(\neg \phi\)’ and ‘\(\neg \psi\)’ true \([TC \neg]\). Let \(\square\) be ‘\(\neg \phi\)’ and \(\triangle\) be ‘\(\neg \psi\)’. Then \(\square\) and \(\triangle\) are both true on I. Since ‘\((\phi \leftrightarrow \psi)\)’ is true on I, and \(\phi\) and \(\psi\) are both false on I, it follows that ‘\(\neg \phi \notin B_s\) and ‘\(\neg \psi \notin B_s\) (otherwise, ‘\((\phi \leftrightarrow \psi)\)’ would be false on I). Hence, \(T_{k+1}[\Sigma]\) is open because only ‘\(\phi\)’ or ‘\(\psi\)’ being on B \((\in B_s)\) would close B.

Since this exhausts the subcases, it follows that \(\square\) and \(\triangle\) are both true on I and that \(T_{k+1}[\Sigma]\) is open.
Case 6: \(\neg (\phi \land \psi)\) on level \(n \leq k\) and we apply the negated conjunction rule to get to level \(k + 1\).

\[
\triangle \text{ on level } k + 1 \text{ [What } \triangle \text{ is depends on why the negated conjunction is true on I.]} 
\]

PROOF LEFT AS HOMEWORK, i.e., you must complete the Subcases of this Case! [Hint: use Case 3 as a model.]

Case 7: \(\neg (\phi \lor \psi)\) on level \(n \leq k\) and we apply the negated disjunction rule to get to level \(k + 1\)

\[
\neg \phi \text{ on level } k + 1 \text{ AND} \\
\neg \psi \text{ on level } k + 1 
\]

Since \(\neg (\phi \lor \psi)\) is on level \(n \leq k\), it is true on I (by IH). So \(\neg (\phi \lor \psi)\) is false on I [TC \(\neg\)], making both \(\neg \phi\) and \(\neg \psi\) false on I [TC \(\lor\)]. But then both \(\neg \phi\) and \(\neg \psi\) are true on I [TC \(\neg\)]. So both sentences at level \(k + 1\) are true on I. Moreover, since both \(\neg \phi\) and \(\neg \psi\) are false on I, \(\phi \notin B_s\) and \(\psi \notin B_s\). But only \(\phi\) or \(\psi\) being on \(B (\subseteq B_s)\) could close B. So, \(T_{k+1}[\Sigma]\) is open.

Case 8: \(\neg (\phi \rightarrow \psi)\) on level \(n \leq k\) and we apply the negated conditional rule to get to level \(k + 1\)

\[
\phi \text{ on level } k + 1 \text{ AND} \\
\neg \psi \text{ on level } k + 1 
\]

Since \(\neg (\phi \rightarrow \psi)\) is true on I (by IH), \(\neg (\phi \rightarrow \psi)\) is false on I [TC \(\neg\)]. So, \(\phi\) is true on I and \(\psi\) is false on I [TC \(\rightarrow\)]. Thus, \(\neg \psi\) is true on I [TC \(\neg\)]. So, both \(\phi\) and \(\neg \psi\) are true on I. Moreover, since \(\phi\) is true on I, \(\neg \phi\) is false on I [TC \(\neg\)]. So, \(\neg \phi \notin B_s\) (since, by IH, everything in \(B_s\) is true on I). And since \(\neg \psi\) is false on I, \(\psi \notin B_s\). But only \(\neg \phi\) or \(\neg \psi\) being on \(B (\subseteq B_s)\) could close B. Hence, \(T_{k+1}[\Sigma]\) is open.

Case 9: \(\neg (\phi \leftrightarrow \psi)\) on level \(n \leq k\); we apply the negated biconditional rule to get to level \(k + 1\)

\[
\square \text{ on level } k + 1 \text{ AND} \\
\triangle \text{ on level } k + 1 \text{ [What } \square \text{ and } \triangle \text{ are depends on why the negated biconditional is true on I]} 
\]

PROOF LEFT AS HOMEWORK.

Case 10: \(\forall x \phi(x)\) on level \(n \leq k\), and we apply the universal rule to get to level \(k + 1\)

\[
\phi(m) \text{ on level } k + 1. 
\]

Since \(\forall x \phi(x)\) is true on I, every \(\alpha \in D\) satisfies \(\phi(x)\) on I [TV \(\forall\)]. Since \(\alpha\) is unrestricted, then no matter what element of D \(m\) designates on I, \(\phi(m)\) is true on I [Def Sat]. So, \(\neg \phi(m)\) is false on I [TC \(\neg\)], and thus \(\neg \phi(m) \notin B_s\). But only \(\neg \phi(m)\) being on \(B (\subseteq B_s)\) could close B. So, \(T_{k+1}[\Sigma]\) remains open.
Case 11: \(\exists x \phi(x)\) on level \(n \leq k\), and we apply the existential rule to get to level \(k + 1\):

\[\phi(m)\] is on level \(k + 1\), and \(\mathfrak{m}\) is new to B.

Let \(\mathfrak{m}\) designate some \(\alpha \in D\) on I, and let \(\alpha\) satisfy \(\phi(x)\) on I. That is, make I so that it makes \(\exists x \phi(x)\) true [TC \(\exists\)] in a way that is useful to us. Since \(\phi(m)\) is true on I, \(\neg \phi(m)\) is false on I [TC \(\neg\)], and so \(\neg \phi(m) \notin B_s\). [Plus, since \(\mathfrak{m}\) is new to the branch, \(\neg \phi(m)\) couldn’t occur earlier on the branch B.] But then \(T_{k+1}[\Sigma]\) is open, since only \(\neg \phi(m)\) being on B (\(\in B_s\)) could close B.

Case 12: \(\neg \forall x \phi(x)\) on level \(n \leq k\), and we apply the negated universal rule to get to level \(k + 1\):

\[\neg \phi(m)\] on level \(k + 1\), and \(\mathfrak{m}\) is new to B

PROOF LEFT AS HOMEWORK. [Hint: use Case 11 as a model.]

Case 13: \(\neg \exists x \phi(x)\) is on level \(n \leq k\) and we apply the negated existential rule to get to level \(k + 1\):

\[\neg \phi(m)\] is on level \(k + 1\)

Since \(\neg \exists x \phi(x)\) is true on I, \(\exists x \phi(x)\) is false on I [TC \(\neg\)]. So, for every \(\alpha \in D\), \(\alpha\) does not satisfy \(\phi(x)\) on I [TC \(\exists\)]. Thus, \(\alpha\) satisfies \(\neg \phi(x)\). Since \(\alpha\) is unrestricted, then no matter what element of D \(\mathfrak{m}\) designates on I, \(\neg \phi(m)\) is true on I [Def Sat]. Moreover, since \(\neg \phi(m)\) is true on I, \(\phi(m)\) is false on I [TC \(\neg\)], so \(\phi(m) \notin B_s\). Only \(\phi(m)\) being on B (\(\in B_s\)) could close B, so, \(T_{k+1}[\Sigma]\) is open.

Case 14: \(\phi(\nu)\) on level \(n \leq k\)

\[(\mathfrak{m} = \mathfrak{m}) / (\mathfrak{m} = \nu)\] on some other level \(n \leq k\)

\[\phi(m)\] is on level \(k + 1\)

Since \(\phi(\nu)\) is true on I (by IH, it is in \(B_s\), since it is on B at some level \(n \leq k\)), for some \(\alpha \in D\), I designates \(\alpha\) by \(\mathfrak{m}\), and \(\alpha\) satisfies \(\phi(x)\) on I [where \(\phi(x)\) is obtained from \(\phi(\nu)\) by replacing at least one \(\mathfrak{m}\) in \(\phi(\nu)\) with \(\mathfrak{x}\)] [Def Sat]. Since \(\phi(\nu) = \mathfrak{m}\) [alternatively, \(\mathfrak{m} = \mathfrak{m}\)] is at level \(n \leq k\), by IH it is also true on I. Thus, I designates \(\alpha\) by \(\mathfrak{m}\) as well [TC =]. Since \(\alpha\) satisfies \(\phi(x)\) on I, \(\phi(m)\) is also true on I. Thus, \(\neg \phi(m)\) is false on I, and \(\neg \phi(m) \notin B_s\). So, \(T_{k+1}[\Sigma]\) is open, since only \(\neg \phi(m)\) being on B (\(\in B_s\)) could close B.

Case 15: \((\mathfrak{m} \neq \mathfrak{m})\) on level \(n \leq k\)

This is not a possible Case because no such sentence can occur at any level \(n \leq k\). By IH, I makes true all of the sentences in set \(B_s\), up to and including that from B at level \(k\). But every instance of \((\mathfrak{m} \neq \mathfrak{m})\) is false on every interpretation, so \((\mathfrak{m} \neq \mathfrak{m}) \notin B_s\) at any level of B.
Once the three Cases left as homework are completed, the result holding for all Cases leads to
the following claim: the sentences at level $k + 1$ are true on I, and $T_{k+1}[\Sigma]$ is open.

**Ladder Principle**: If $T_n[\Sigma]$ is open at every $|B| = n \leq k$, and some I makes all the member of $B_\Sigma$ true, up to and including that from level $k$, then the sentence(s) at level $k + 1$ is/are true on I, and $T_{k+1}[\Sigma]$ is open.

**Conclusion**: Since $T_0[\Sigma]$ is open and some I makes $\Sigma$ true (Basis), it follows, from the conditional Ladder Principle, that all $T_n[\Sigma]$ are open—including every completed $T[\Sigma]$—and some I makes all the sentences of $B_\Sigma$ true.

That ends the mathematical induction. But now, since we assumed that $\Sigma$ was consistent, and we showed that, on that assumption, it follows that all of $\Sigma$’s trees are open, we have proven the original (conditional) claim of DAT.

**Remember to do the homework problems!**