Applying External Interpretations to Quantified Sentences

Consider the following interpretation, I, for some expressions of FOL:

The domain of discourse, D, is the set of objects \{1, 2, 3, 4\}.
The name ‘a’ denotes 1.
The name ‘b’ denotes 2.
The name ‘c’ denotes 3.
The name ‘d’ denotes 4.
The following predicates have the following extensions.
F: \{1, 2, 3\}
G: \{2, 3, 4\}
R: \langle1, 1\rangle, \langle2, 2\rangle, \langle3, 3\rangle, \langle4, 4\rangle
S: \langle1, 2\rangle, \langle2, 3\rangle, \langle3, 4\rangle

Here is how to determine the truth-values of an FOL sentence on interpretation I, by employing the notion of objects (or ordered pairs, or larger n-tuples, of objects) satisfying open wffs of FOL and the truth-tables for the truth-functional connectives.

I. Consider the sentence ‘\(\forall x(R(b, x) \lor S(x, b))\)’. For the sentence to be true, every object in the domain, D, must satisfy the open formula bound by the universal quantifier. That holds only if substituting the name of each object from D in for ‘x’ in the open wff bound by the quantifier yields a true sentence. The substitutions yield the following sentences.

1. \(R(b, a) \lor S(a, b)\)
2. \(R(b, b) \lor S(b, b)\)
3. \(R(b, c) \lor S(c, b)\)
4. \(R(b, d) \lor S(d, b)\)

The first is true on I because ‘\(S(a, b)\)’ is true on I, given that ‘a’ denotes 1, ‘b’ denotes 2 and \langle1, 2\rangle is in the extension of ‘\(S\)’. The second one is true on I because ‘\(R(b, b)\)’ is true on I, given that ‘b’ denotes 2 and \langle2, 2\rangle is in the extension of ‘\(R\)’. The third is false on I, since both disjuncts, ‘\(R(b, c)\)’ and ‘\(S(c, b)\)’, are false on I—the first because ‘b’ denotes 2 and ‘c’ denotes 3, and \langle2, 3\rangle is not in the extension of ‘\(R\)’; the second because \langle3, 2\rangle is not in the extension of ‘\(S\)’. The fourth sentence is false on I, since both of its disjuncts, ‘\(R(b, d)\)’ and ‘\(S(d, b)\)’, are false on I—the first because ‘b’ denotes 2 and ‘d’ denotes 4, and \langle2, 4\rangle is not in the extension of ‘\(R\)’; the second because \langle4, 2\rangle is not in the extension of ‘\(S\)’. Since substituting the names ‘c’ and ‘d’ in for ‘x’ in the open wff bound by the quantifier yields false sentences, the wff is not satisfied by the objects from the domain, D, that these names denote on I. Thus, since not every object from the domain satisfies the open wff the quantifier binds, the universal sentence (1) is false on I.

[Actually, recall that all it would really take to show the whole sentence is false on I is finding just one object from D that doesn’t satisfy the open wff.]

II. Now consider the sentence ‘\(\exists y \forall x S(y, x)\)’. For this sentence to be true on I, there must be at least one object from D that satisfies the open wff bound by the existential quantifier. That holds so long as there is at least one name for an object from D such that substituting that name in for ‘y’ in the open wff bound by the quantifier yields a true sentence. The possible substitutions yield the following sentences.
1. \( \forall x S(a, x) \)
2. \( \forall x S(b, x) \)
3. \( \forall x S(c, x) \)
4. \( \forall x S(d, x) \)

So long as just one of these “substitution-instance” sentences is true on I, the whole sentence will be true on I. Each of these “substitution-instance” sentences is a universal claim. A universal claim is true on I only if every object in D satisfies the open wff bound by the universal quantifier. That holds for each of 1-4 only if all substitutions of a name of an object from D in for ‘x’ in the open wff yields a true sentence. So, for each of 1-4, there are four further “substitution-instances”. These further substitutions yield the following sentences.

1. i. \( S(a, a) \)
   ii. \( S(a, b) \)
   iii. \( S(a, c) \)
   iv. \( S(a, d) \)
2. i. \( S(b, a) \)
   ii. \( S(b, b) \)
   iii. \( S(b, c) \)
   iv. \( S(b, d) \)
3. i. \( S(c, a) \)
   ii. \( S(c, b) \)
   iii. \( S(c, c) \)
   iv. \( S(c, d) \)
4. i. \( S(d, a) \)
   ii. \( S(d, b) \)
   iii. \( S(d, c) \)
   iv. \( S(d, d) \)

For “substitution-instance” sentence 1 to be true on I, all of 1.i-1.iv. must be true on I. Given what objects from D the names denote on I, for 1.i. to be true on I, the ordered-pair \(<1, 1> \) must be in the extension of ‘S’. It is not, so 1.i. is false. That is enough to make sentence 1 false on I. So now we have to check sentence 2, since the whole sentence being true on I just requires one of 1-4 being true on I. For sentence 2 to be true on I, all of 2.i-2.iv must be true on I. Given what objects from D the names denote on I, for 2.i to be true on I, the ordered pair \(<2, 1> \) must be in the extension of ‘S’. It is not, so 2.i is false on I. That is enough to make sentence 2 false on I. So, now we have to check sentence 3. For sentence 3 to be true on I, all of 3.i-3.iv must be true on I. Given what objects from D the names denote on I, for 3.i to be true on I, the ordered pair \(<3, 1> \) must be in the extension of ‘S’. It is not, so 3.i is false on I. That is enough to make sentence 3 false on I. So, now we have to check sentence 4. For sentence 4 to be true on I, all of 4.i-4.iv must be true on I. Given what objects from D the names denote on I, for 4.i to be true on I, the ordered pair \(<4, 1> \) must be in the extension of ‘S’. It is not, so 4.i is false on I. That is enough to make sentence 4 false on I. So, all of 1-4 are false on I. That makes the whole sentence ‘\( \exists y \forall x S(y, x) \)’ false on I.
For practice, see if you can determine the truth-values of the following sentences, using the above method.

III. \( \exists x S(x, b) \)
IV. \( \forall x (F(x) \rightarrow G(x)) \)
V. \( \exists x (R(x, x) \land \neg \exists y S(x, y)) \)
VI. \( \forall x (F(x) \leftrightarrow \exists y S(x, y)) \)
VII. \( \forall x (\exists y S(x, y) \rightarrow \exists z R(x, z)) \)

You might also try your hand at constructing interpretations by constructing for each of III-VII independently, another interpretation on which the sentence under consideration has the opposite truth-value from what is has on I.