Review for the final exam (Math 127)

1. Evaluate
   (a) \( \left( \tan\left(\frac{13\pi}{4}\right) - \tan\left(-\frac{11\pi}{3}\right) \right)^2 \)
   (b) \( \left( \cos\left(-\frac{\pi}{4}\right) + \cos(10\pi) \right)^2 \)
   (c) \( \frac{1}{\sec\left(\frac{7\pi}{4}\right) - \cos\left(\frac{3\pi}{4}\right)} \)
   (d) \( \cos^{-1}\left(-\frac{1}{2}\right) + \tan^{-1}\left(-\sqrt{3}\right) \)

2. (a) If \( \sec\theta = 2 \), and \( 270^\circ < \theta < 360^\circ \), find \( \cos\theta \), \( \sin\theta \), \( \tan\theta \), \( \cot\theta \), \( \csc\theta \)
   (b) If \( \tan\theta = -2\sqrt{2} \), \( 270^\circ < \theta < 360^\circ \) evaluate \( \cos(2\theta) \), \( \sin(2\theta) \)
   (c) Write the \( \csc\theta \) in terms of \( \cot\theta \) for in the Quadrant III.
   (d) Find the values of all the trigonometric functions of \( t \) from the given information:
      \( \tan t = -3/4; \cos t < 0 \)

3. Determine whether the function is even, odd, or neither
   (a) \( f(x) = \sin x \cos x \)
   (b) \( f(x) = x^2 \cos(2x) \)
   (c) \( f(x) = \sin x + \cos x \)

4. Find the exact value of the expression
   (a) \( \tan\left(\cos^{-1}\frac{1}{2}\right) \); (b) \( \sec\left(\sin^{-1}\frac{12}{13}\right) \); (c) \( \sin^{-1}\left(\sin\frac{4\pi}{5}\right) \); (d) \( \frac{\cos^{-1}\left(\cos\frac{2\pi}{3}\right)}{\sin^{-1}\left(\sin\frac{4\pi}{3}\right)} \)
   (e) \( \cos^{-1}\left(\cos\frac{\pi}{6}\right) \); (f) \( \tan^{-1}\left(\tan\frac{2\pi}{3}\right) \); (g) \( \sin^{-1}\left(\sin\frac{5\pi}{8}\right) \)

5. (a) Express \( \cos\left(2\tan^{-1}\frac{x}{4}\right) \) and \( \sin\left(2\tan^{-1}\frac{x}{4}\right) \) as an algebraic expression in \( x \)
   (b) express \( \sin\left(\tan^{-1} x - \tan^{-1} y\right) \) in terms of \( x \) and \( y \) only
   (c) Write the expression \( \sin\left(\frac{1}{2}\cos^{-1} x\right) \) as an algebraic expression in \( x \)
   (d)

The domain and range for the function \( \cos^{-1}(x) \) are:
The domain and range for the function $\sin^{-1}(x)$ are:

The domain and range for the function $\tan^{-1}(x)$ are:

The domain and range for the function $\cot^{-1}(x)$ are:

(e) Find the exact value of each expression $\sin\left(\cos^{-1}\frac{1}{2} - \tan^{-1}1\right)$

6. (a) Find the length of the arc that subtends an angle of measure $60^\circ$ on a circle of diameter $18\text{cm}$; (b) Find the area of a sector in part (a)

7. (a) The area of the triangle $\Delta ABC$ is $A = 25\sqrt{3}$. If $\angle BAC = 30^\circ$ find $|AC| = |BC| = ?$

(b) Given triangle ABC with sides length $a=6$, $b=6$, $c=4$. Find the area of the triangle.

(c) Find the area of the shaded region in the figure.

(d) An isosceles triangle has an area of $100\text{ cm}^2$, and the angle between two equal sides is $\theta = \frac{5\pi}{6}$. What is the length of the two equal sides?

8.

(a) Simplify $\frac{\cot \theta}{\csc \theta - \sin \theta}$

(b) Simplify $\left(\frac{\sin x}{1 + \cos x} - \frac{\sin x}{\cos x - 1}\right)^{-1}$

(c) Simplify $\tan^4 t + \frac{\sin t}{\cos t}$

(d) Simplify $\frac{1 + \cot^2 x}{1 + \tan^2 x}$

(e) Simplify $\left(\frac{\cot x}{1 - \csc x} - \frac{1 - \csc x}{\cot x}\right)^{-1}$

(f) Prove $\frac{1-\cos x}{1+\cos x} - \frac{1+\cos x}{1-\cos x} = -4\cot x \csc x$
(g) Prove \( \frac{\tan^2 x}{\sec x - 1} = \frac{1 + \cos x}{\cos x} \)

(h) Prove \( (\sin x + \cos x)^2 = 1 + \sin 2x \)

(i) Prove \( \frac{2(\tan x - \cot x)}{\tan^2 x - \cot^2 x} = \sin 2x \)

9. Evaluate
   (a) \( \sin 75^\circ \);   (b) \( \cos 75^\circ \);   (c) \( \cos \left( \frac{\pi}{12} \right) \);   (d) \( \sin \left( \frac{\pi}{12} \right) \);   (e) \( \sin(22.5^\circ) \);   (f) \( \cos(67.5^\circ) \)

10. Given that \( \sin x = \frac{5}{13} \), \( x \) is in Quadrant I, and \( \cos y = -\frac{2\sqrt{5}}{5} \), \( y \) is in the Quadrant III, calculate
    (a) \( \cos(x - y) \);   (b) \( \cos(x + y) \);   (c) \( \sin(x - y) \);   (d) \( \sin(x + y) \);   (e) \( \cos(2x) \);   (f) \( \cos(x/2) \)

11. Given that \( \csc x = 3 \), \( 90^\circ < x < 180^\circ \), find \( \sin \left( \frac{x}{2} \right) \), \( \cos \left( \frac{x}{2} \right) \), \( \tan \left( \frac{x}{2} \right) \)

12. Find the value
    \( \cos 255^\circ - \cos 195^\circ = \)
    \( \cos 255^\circ + \cos 195^\circ = \)
    \( \sin 255^\circ + \sin 195^\circ = \)
    \( \sin 255^\circ - \sin 195^\circ = \)

13. Solve the given equation
    (a) \( \sec x (2 \sin x - \sqrt{2}) = 0 \)
    (b) \( \sqrt{3} \tan 3x + 1 = 0 \)
    (c) \( \cos(2x) + \cos x = 2, \quad x \in [0, 2\pi) \)
    (d) \( 4 \sin^2 x + 4 \cos x = 1, \quad x \in [0, 2\pi) \)
    (e) \( 2 \cos^2 x - 5 \sin x - 4 = 0, \quad x \in [0, 2\pi) \)
14. Find the sum of the series:

(a) \[ \sum_{k=1}^{\infty} \left( \frac{1}{\sqrt{2}} \right)^k \]

(b) \[ \sum_{k=0}^{\infty} \left( \frac{1}{\sqrt{2}} \right)^k \]

(c) \[ \frac{4}{9} - \frac{8}{27} + \frac{16}{81} - \frac{32}{243} + \ldots \]

15. The 8th term of arithmetic sequence is 16, and the 18th term is 56. Find the sum of the first 20 terms. Find \( a_{11} + a_{12} + \ldots + a_{20} \).

16. A person gets a job with a salary of $22,000 a year. He promised a $450 raise each subsequent year. Find his total earning for 21-year period.

17. (a) Write \( z = 2 \left( \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right) \) in rectangular form.

(b) Write \( z_1 = 2\sqrt{3} - 2i \) and \( z_2 = -1 + i \) in polar form, and then find the product \( z_1z_2 \), and the quotients \( \frac{z_1}{z_2} \) and \( \frac{1}{z_1} \).

18. Find the indicated power using De Moivre’s Theorem \( \left( \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i \right)^{12} \)

19. Find the square roots of \( 4\sqrt{3} + 4i \). Graph roots in the complex plane.

20. (a) Find the angle between vectors \( \vec{w} = -3 \hat{i} + \sqrt{3} \hat{j} \) and \( \vec{v} = \left\langle \sqrt{3}, -3 \right\rangle \)

(b) Let \( \theta \) be an angle between vectors \( \vec{w} = \hat{i} + 3 \hat{j} \) and \( \vec{v} = \left\langle 5, 2 \right\rangle \). What is \( \cos \theta \)? What is \( \sin \theta \)?

21. A constant force \( \vec{F} = \left\langle 3, \sqrt{3} \right\rangle \) moves an object along a straight line from the point \( (0,1) \) to the point \( (\sqrt{3},4) \). Find the distance \( \vec{D} \). Find the work done if the distance is measured in feet, and the force is measured in pounds. Find the cosine of an angle between the force \( \vec{F} \) and the distance \( \vec{D} \).

22. From a point C on the ground level, the angle of elevation to the top of a tree is 30°. From a point D, which is closer to the tree, the angle of elevation is measured to be 45°. Find the distance between points C and D if the height of the tree is 4 m.

23. (a) A 20-ft ladder leans against a building. The angle between the ground and the ladder is 60°. How high does the ladder reach on the building?
(b) From the top of a 200-ft lighthouse, the angle of depression to a ship in the ocean is 30°. How far is the ship from the base of the lighthouse?

(c) 600-ft guy wire is attached to the top of a communications tower. If the wire makes an angle of 60° with the ground, how tall is the communications tower?

(d) An airplane is flying at an elevation of 5100 ft. directly above a straight highway. Two motorists are driving cars on the highway on opposite sides of the plane and the angle of depression to one car is 30° and to the other car is 60°. How far apart are the two cars?

(e) An airplane is flying at an elevation of 9000 feet, directly above a straight highway. Two motorists are driving cars on the highway, both on one side of the plane. If the angle of depression to one car is 60° and to the other is 45°, how far apart are the cars?

24. The law of cosines:

(a) Two boats leave the same port at the same time. One travels at a speed 40 mi/h in the direction N 48° E and the other travels at a speed of 20 mi/h in a direction S 12° E. How far apart are the two boats after 3/2 hour?

(b) Two boats leave the same port at the same time. One travels at a speed 10 mi/h in the direction S 12° W and the other travels at a speed of 20 mi/h in a direction S 72° W. How far apart are the two boats after one hour?

(c) A pilot flies in a straight path for 2 hours. She then makes a course correction, heading 45° to the right of her original course, and flies 1 hour in the new direction. If she maintains a constant speed of 200 mile per hour, how far is she from her starting position?

(d) Two straight roads diverge at an angle 45°. Two cars leave the intersection at 4:00 P.M., one travelling at 40 mi/h, and the other at 60 mi/h. How far apart are the cars at 4:30 P.M.?

(e) Use either the Law of sines or the Law of cosines to determine possibilities for angle \( b = 10\sqrt{13}, \ a = 10, \ c = 30 \).

(f) Triangle ABC has \( a = 7, \ b = 8 \) and \( c = 10 \). Solve for \( \cos A \).

25. The law of sines:

(a) A satellite orbiting the Earth passes directly overhead at observation stations in Phoenix and Los Angeles, 340 mi apart. At an instant when the satellite is between these two stations, its angle of elevation observed at Los Angeles is 60°. If the distance from the satellite to Phoenix is known to be 170\(\sqrt{6}\) miles, find the angle of elevation observed at Phoenix.

(b) To find the distance across a river, a surveyor chooses points A and B, which are 280 ft apart on one side of the river. She then chooses a reference point C on the opposite side of the river and finds that \( \angle BAC = 60° \) and \( \angle ABC = 75° \). Find the distance from B to C?

(c) Assume you have a triangle ABC with \( a = 5, \ b = 5\sqrt{2}, \) and angle opposite the side a, is \( \angle A = 30° \). Find B (the angle opposite the side b).

How many solutions do you have?
(d) A fire is sighted from stations A and B. The bearing of the fire from station A is S 60° E, and the bearing from the station B is S 30° W. Station B is 4 miles due east of station A. How far is the fire from station A?

(e) Use the Law of sines to solve for the missing values of the triangle which satisfy the given conditions below. (Determine if there are 0, 1, or 2 triangles): \( a = 15\sqrt{3}, \ b = 15, \ \angle B = 30^\circ \).

(f) Use the Law of sines to solve for the missing values of the triangle which satisfy the given conditions below. (Determine if there are 0, 1, or 2 triangles): \( a = 20, \ b = 40, \ \angle B = 135^\circ \).

26.

I. A straight river flows east at a speed of 20mi/h. A boater starts at the south shore of the river and heads in the direction which is 60° to the south shore. The motorboat has a speed 20mi/h relative to the water.

(a) Express the velocity of the river as a vector in component form;
(b) Express the velocity of the motorboat relative to the water as a vector in component form;
(c) Find the true velocity of the motorboat;
(d) Find the true speed and direction of the motorboat.

II. A migrating salmon heads in the direction 0 N45 E, swimming at 5 mi/h relative to the water.

The prevailing ocean currents flow due east at 3 mi/h. Find the true velocity of the fish as a vector.

III. A straight river flows east at a speed of 20 mph. A boater starts at the south shore of the river and heads in a direction 60° from the shore (see figure in book). The motorboat has a speed of 20 mph relative to the water.

(a) Express the velocity of the river as a vector in component form.
(b) Express the velocity of the motorboat relative to the water as a vector in component form.
(c) Find the true velocity of the motorboat
(d) Find the true speed and direction of the motorboat.

27. Find the amplitude, period, and phase shift of the function, and graph one complete period

(a) \( y = 5\cos\left(3x - \frac{\pi}{4}\right) \);
(b) \( y = 2\sin\left(\frac{2}{3}x - \frac{\pi}{6}\right) \)

(c) \( y = -3\cos\left(2x - \frac{2\pi}{3}\right) \);
(d) \( y = 3\sin\left(2x - \frac{2\pi}{3}\right) \)

(e) \( y = 4\cos\left(x + \frac{\pi}{3}\right) \);
(f) \( y = 4\sin\left(x + \frac{\pi}{3}\right) \)

28. Find the period and phase shift, identify the asymptotes and graph
(a) \( y = 5 \sec \left( 3x - \frac{\pi}{2} \right) \);
(b) \( y = -\tan \left( 2x - \frac{\pi}{3} \right) \)
(c) \( y = 5 \csc \left( 3x - \frac{\pi}{2} \right) \);
(d) \( y = \cot \left( 2x - \frac{\pi}{3} \right) \)
(e) \( y = 2 \sec \left( \frac{x}{2} \right) \);
(f) \( y = 2 \tan \left( \frac{x}{2} \right) \)

29. Sketch a graph of the polar equation

(a) \( r = 2 - 2 \cos \theta \);
(b) \( r = \sin(2\theta) \);
(c) \( r = 2 \cos 3\theta \)
(d) \( r = 1 - 2 \cos \theta \);
(e) \( r = 1 - 2 \sin \theta \)

30. A pair of parametric equations is given. Find a rectangular-coordinate equation for the curve by eliminating the parameter. Sketch the graph.

(a) \( x = \sec t, \quad y = \tan^2 t, \quad 0 \leq t < \frac{\pi}{2} \)
(b) \( x = \cos t, \quad y = \cos(2t), \)
(c) \( x = t^2, \quad y = t - 2, \quad 2 \leq t \leq 4 \)
(d) \( x = 3 \cos t, \quad y = 3 \sin t, \quad 0 \leq t \leq \pi \)
(e) \( x = 3 \sin t, \quad y = 3 \cos t, \quad 0 \leq t \leq \pi \)

31. Complete the square to determine whether the equation represents an ellipse, a parabola, a hyperbola or a degenerate conic. For ellipse: find center, foci, vertices, and length of the major and minor axes; For parabola: find the vertex, focus, and directrix; For hyperbola: find the center, foci, vertices, and asymptotes. Sketch the graph.

(a) \( x^2 + 6x + 12y + 9 = 0 \)
(b) \( 2x^2 + y^2 = 2y + 1 \)
(c) \( 16x^2 - 9y^2 - 96x + 288 = 0 \)
(d) \( x^2 + 16 = 4(y^2 + 2x) \)

32. Find the radius and the center of the circle given by \( 2x^2 + 8x + 2y^2 - 12y + 25 = 0 \)