28 Let \( a \) be a point on a (directed) curve \( K \) having Schwarz function \( S(z) \).

(i) Show that the curvature of \( K \) at \( a \) is

\[
\kappa \equiv \dot{\phi} = \frac{i}{2} \frac{S''(a)}{|S'(a)|^{3/2}},
\]

where \( \phi \) is the angle in [30], and the dot denotes differentiation with respect to distance / along \( K \) (in the given sense).

With the hints, it is given that \( \dot{\phi} = \frac{dl}{d\phi} \), \( S'' = \frac{dS'}{dz} \), \( dz = e^{i\phi} dl \), and \( S' = e^{-i2\phi} \).

Calculate \( S'' \) and \( |S'(a)|^{3/2} \). Then try to reduce the RHS to an equation containing \( \frac{d\phi}{dl} \) because \( \dot{\phi} = \frac{d\phi}{dl} \).

\[
S'' = \frac{dS'}{dz} = -i2e^{-i2\phi} \frac{d\phi}{dz}
\]

\[
|S'(a)|^{3/2} = (e^{-i2\phi})^{3/2} = e^{-i3\phi}
\]

\[
\frac{i}{2} \frac{S''(a)}{|S'(a)|^{3/2}} = \frac{i}{2} \frac{-i2e^{-i2\phi} \frac{d\phi}{dz}}{e^{-i3\phi}} = \frac{\phi}{d\phi/dz} = \dot{\phi} = \kappa
\]

Deduce that

\[
|\kappa| = |S''/2|
\]

\[
|S''/2| = |(-i2e^{-i2\phi} \frac{d\phi}{dz})/2|
\]

\[
= |d\phi/dz|
\]

\[
= |d\phi/e^{i\phi} dl|
\]

\[
= |d\phi/dl| = |\dot{\phi}| = |\kappa|
\]

(ii) Deduce that the centre of curvature of \( K \) at \( a \) is \( \{ a + 2S'(a)/S''(a) \} \).

By [30], \( \phi \) is the angle between the tangent and the real line. Let \( q \) be the center of the circle of curvature. We can write

\[
a = q - |a - q| e^{i\phi} \quad \text{(minus sign because } ie^{i\phi} \text{ points at } q \text{ from } a)\]

\[
q = a + rie^{i\phi}
\]
\[ a + \frac{1}{k} ie^{i\phi} \]

\[ = a + i[S'(a)]^{-\frac{1}{2}} \left( \frac{1}{2} \left( (S''(a))/[S'(a)]^{3/2} \right) \right) \]  

(see (iii) for derivations)

\[ = a + \frac{2S(a)}{S'(a)} \]

(iii) Show that the rate of change of the curvature of \( K \) is given by the “Schwarzian derivative” [Ex. 19] of the Schwarz function:

\[ \kappa' = \frac{1}{2S} \{ S(z), z \} \]

The key is to recognize that \( \kappa' \) is \( \frac{d\kappa}{dl} \), not \( \frac{d\kappa}{dz} \).

\[ \kappa = \frac{i}{2} \frac{S''}{[S']^{3/2}}, \quad \text{from (i)} \]

\[ \frac{d\kappa}{dz} = \frac{i}{2} \left( S''/[S']^{3/2} \right)' \]

\[ \frac{d\kappa}{dl} \frac{dl}{dz} = \frac{i}{2} \left( S''[S']^{-3/2} \right)' \]

\[ \frac{d\kappa}{dl} = \frac{i}{2} \frac{dz}{dl} \left[ (S'')' [S']^{-3/2} + S'' \left( [S']^{-3/2} \right)' \right] \]

\[ S'(a) = e^{-i2\phi}, \text{ given} \]

\[ S'(a)^{1/2} = e^{-i\phi} \]

\[ e^{i\phi} = S'(a)^{-1/2} \]

\[ \kappa = \frac{1}{2} e^{i\phi} \left( (S'')' [S']^{-3/2} - (3/2) S''[S']^{-5/2} S'' \right), \text{ because } \frac{dz}{dl} = e^{i\phi} \]

\[ = \frac{1}{2} S^{(-1/2)} \left[ (S'')' /[S']^{3/2} - (3/2)(S'')^2/[S']^{5/2} \right] \]

\[ = (iS^{(-1/2)}/2 \left( S'\right)^{1/2}) \left( \frac{S''}{S} \right)^{1/2} \left( \frac{S''}{S} \right)^{1/2} \]