19. In more advanced work on Möbius transformations (e.g., Nehari [1952] and Beardon [1984]), an important role is played by the so-called Schwarzian derivative \( \{f(z), z\} \) of an analytic function \( f(z) \) with respect to \( z \):

\[
\{f(z), z\} \equiv \left( \frac{f''}{f'} \right)' - \frac{1}{2} \left( \frac{f''}{f'} \right)^2
\]

(i) Show that the Schwarzian derivative may also be written as

\[
\{f(z), z\} = \frac{f'''}{f'} - \frac{3}{2} \left( \frac{f''}{f'} \right)^2
\]

Answer:

Use Leibnitz' quotient rule to find \( \left( \frac{f''}{f'} \right)' \).

\[
\left( \frac{f''}{f'} \right)' = \frac{f'''f' - f''f''}{(f')^2}
\]

\[
= \frac{f'''f' - (f')^2}{(f')^2}
\]

Substitute (1) into \( \{f(z), z\} \).

\[
\{f(z), z\} \equiv \frac{f'''f' - (f')^2}{(f')^2} - \frac{1}{2} \left( \frac{f''}{f'} \right)^2
\]

\[
= \frac{f'''f' - (f')^2}{(f')^2} - \frac{1}{4} \left( \frac{f''}{f'} \right)^2
\]

\[
= \frac{f'''f' - (3/2)(f')^2}{(f')^2}
\]

\[
= \frac{f'''}{f'} - \frac{3}{2} \left( \frac{f''}{f'} \right)^2
\]

QED

(ii) Show that \( \{az + b, z\} = 0 = \{(1/z), z\} \).

\[
\{f(z), z\} = \frac{f'''}{f'} - \frac{3}{2} \left( \frac{f''}{f'} \right)^2 \quad \text{given}
\]

\[
\{az + b, z\} = \frac{(az+b)''}{(az+b)'} - \frac{3}{2} \left( \frac{(az+b)''}{(az+b)'} \right)^2 \quad \text{by substitution into } \{f(z), z\}
\]
\[
\frac{\frac{\partial z}{\partial z} \cdot (az + b)'}{\frac{\partial z}{\partial z} \cdot (az + b)} = 0
\]

Hence,

\[\{az + b, z\} = 0\]

Show that \(\{(1/z), z\} = 0\).

\[f' = (z^{-1})' = -z^{-2}\]
\[f'' = (-z^{-2})' = 2z^{-3}\]
\[f''' = (2z^{-3})' = -6z^{-4}\]

\[\{(1/z), z\} = \frac{-6z^{-4}}{-z^{-2}} - \frac{3}{2}\left(\frac{2z^{-3}}{-z^{-2}}\right)^2\]
\[= 6z^{-2} - \frac{3}{2}\left(-2z^{-1}\right)^2\]
\[= 6z^{-2} - \frac{3}{2}\left(4z^{-2}\right)\]
\[= 6z^{-2} - 6z^{-2} = 0\]

QED

(III) Let \(f\) and \(g\) be analytic functions, and write \(w = f(z)\). Show that the Schwarzian derivative of the composite function \(g[f(z)] = g[w]\) is given by the following “chain rule”:

\[\{g(w), z\} = \left[f'(z)\right]^2\{g(w), w\} + \{f(z), z\}\]

Answer:

It will be convenient to let \(g = g(w)\) and \(w = f\) and assume \(f = f(z)\).

\[g'(w) = gf'\]
\[g''(w) = g''f^2 + g'f'' = g''(f')^2 + g'f''\]
\[g'''(w) = (g''(f')^2 f' + g''2f') + (g'f'f' + g'f'' f')\]
\[= g''(f')^2 + 3g''f'f' + g'f''\]
\[
g' - g' = \frac{g''(f')^3 + 3g'f'f'' + g'''}{g'f'}
\]
\[
= \frac{(f')^2 g''}{g'} + \frac{2g'' f'}{g'} + \frac{f''}{f'}
\]

\[
\left( \frac{g'}{g} \right)^2 = \frac{(g'(f')^2 + g'' f')^2}{(g'(f')^3)^2}
\]
\[
= \frac{(g'(f')^4 + 2g'(f')^2 g' f'' + (g')^2 (f')^2)}{(g'(f')^3)^2}
\]
\[
= (f')^2 \left( \frac{g'}{g} \right)^2 + \frac{2g' f'}{g'} + \left( \frac{f''}{f'} \right)^2
\]

\[
\{g(w), z\} = \frac{g''}{g} - \frac{3}{2} \left( \frac{g'''}{g} \right)^2 = \left[ \frac{(f')^2 g''}{g'} + \frac{3g' f''}{g'} + \frac{f''}{f'} \right]
\]
\[
- \frac{3}{2} (f')^2 \left( \frac{g'}{g} \right)^2 + \left( \frac{f''}{f'} \right)^2
\]
\[
= \frac{(f')^2 g''}{g'} - \frac{3}{2} (f')^2 \left( \frac{g'}{g} \right)^2 + \frac{f''}{f'} - \frac{3}{2} \left( \frac{f''}{f'} \right)^2
\]
\[
= (f')^2 \left( \frac{g'}{g} - \frac{3}{2} \left( \frac{g'}{g} \right)^2 \right) + \left( \frac{f''}{f'} - \frac{3}{2} \left( \frac{f''}{f'} \right)^2 \right)
\]
\[
= (f')^2 \{g(w), w\} + \{f(z), z\}
\]

QED

(iv) Use the previous two parts to show that all Möbius transformations have vanishing Schwarzian derivative. (Hint: Recall that the mappings in part (ii) generate (via composition) the set of all Möbius transformations.)

\[
\{az + b, z\} = 0 = \{(1/z), z\} \quad \text{from (ii)}
\]
\[
\{g(w), z\} = (f')^2 \{g(w), w\} + \{f(z), z\} \quad \text{from (iii)}
\]

Not having seen Vasco’s systematic method of decomposing the transformation using the steps from (3), p. 124, I decided to include step 1 with step 2 and step 4 with step 3 because both 1 and 4 involve adding constants, which gives the format (az + b), for which we know that the Schwarzian is 0.

Let f(z) = z + d/c

\[
\{f(z), z\} = 0 \quad \text{(from (ii), \{az + b, z\} = 0)}
\]
Let \( g(z) = 1/z \). The first step of our revised composition is \( g(f) \).

\[
g(f) = 1/f = 1/(z+d/c)
\]

\( \{g(f), f\} = 0 \quad \text{from (ii), \{1/z, z\} = 0} \)

Let \( h(z) = \frac{(ad-bc)}{c^2} z + a/c \). The second step of the composition is \( h(g) \).

\[
h(g(f)) = -(ad - bc) g/c^2 + a/c
\]

\( \{h(g), g\} = 0 \quad \text{from (ii), \{az + c, z\} = 0} \)

We want the Schwarzian of \( h(g(f(z))) \).

\[
\{h(g), z\} = (g')^2 \{h(g), g\} + \{g(f), z\} \quad \text{(from (iii)) [valid step?]}
\]

\[
\{g(f), z\} = (f')^2 \{g(f), f\} + \{f(z), z\}
\]

\[
\{h(g), z\} = (g')^2 \{h(g), g\} + [(f')^2 \{g(f), f\} + \{f(z), z\}] \quad \text{(by substitution)}
\]

\[
= (g')^2 \ast 0 + (f')^2 \ast 0 + 0
\]

\[
= 0
\]

(v) Use the previous two parts to show that the Schwarzian derivative is “invariant under Möbius transformations”, in the following sense: if \( M \) is a Möbius transformation, and \( f \) is analytic, then

\[
\{M[f(z)], z\} = \{f(z), z\}
\]

\[
\{M[f(z)], z\} = (f')^2 \{M, f\} + \{f, z\}
\]

\[
= (f')^2 \ast 0 + \{f, z\}
\]

\[
= \{f(z), z\} \]