Figure 1. $e^z$ on arbitrary directed line segment $S$
17. Let $S$ be a directed line segment through a point $p = x + iy$.

(i) Let $f(z) = e^z$. Without calculation, decide which direction of $S$ yields an image $f(S)$ having vanishing curvature at $f(p)$.

The method of (i): For the curvature of $f(s)$ (written $\kappa$) to vanish at $p$ (or any other point), $f(S)$ must be a line segment. The solution therefore requires finding the constraints under which $f(S)$ is a line segment.

Rewrite $e^z = e^x e^{iy}$. We can see that $f(S) = e^z$ is a straight line only if $y$ is a constant. Therefore, $p = x + iy$ must lie on a line parallel to the real line. $S$ has the direction $\theta = 0$.

Figure 1 shows a plot of $f(S) = e^z$ on an arbitrary directed line segment $S$. In Figure 2, we pick a point $p$ and draw a line segment $S$ through $p$ parallel to the real line, and replot $f(S)$. Now we see that $\kappa$ has the direction $\arg(i\xi) = \pi/2$ (green arrow) for all $p$ and that there is no projection of $\kappa$ onto $S$.

[Note on the plot itself: In Figure 1, the circle of curvature (dotted gray line) was plotted by calculating the center from three points in the neighborhood of $e^p$. The length of the radius was obtained as $1/(\kappa \cdot \xi)$ and it was checked against a calculation from three points.]
Vasco has rewritten (ii) as follows (Forum, errata for Ch 5, Ex 17):

(ii) The complex curvature $\mathcal{K}$ must therefore point in one of the two directions which are orthogonal to the direction of $S$ obtained in part (i). Which? By considering the curvature of the image of $S$ when $S$ points in the direction you have just decided on for $\mathcal{K}$, deduce the value of $|\mathcal{K}|$, and thereby conclude that $\mathcal{K} = ie^{-x}$.

The method of (ii): There appear to be only two ways to determine the sign of $\mathcal{K}$: (1) by calculation of $f''/f'$, or (2) by inspection of a plot of $S$ and $f(S)$ after $S$ has been rotated around $p$ to be parallel (symbol “$||$”) to $\mathcal{K}$. In method (2), one must trace the direction of movement $f(p)$ when $p$ moves in a particular direction on $S$. But since $\mathcal{K}$ vanishes at $p$, we must first rotate $S$ around $p$ by $\pm \pi/2$ in order to make it parallel to $\mathcal{K}$. We rotate around $p$ because this rotated $S$, which we will label $\tilde{S}$, must still pass through $p$ after the rotation. $\tilde{K}$ is at its maximum value on $\tilde{S}$ and $f(\tilde{S})$ displays maximum curvature. We can plot $f(\tilde{S})$ with arrows revealing its direction of propagation with the movement of $p$ on $\tilde{S}$ at any point.

Suppose that $S$ parallels the real line (as with $e^z$) when $\mathcal{K}$ vanishes at $p$. $\mathcal{K}$ is necessarily orthogonal to $S$, but we must guess whether it points up or down. We guess that it points up. We rotate $S$ until $S \parallel \mathcal{K}$ and move $p$ along $\tilde{S}$ in the direction of $\mathcal{K}$ (now become the direction of increasing $y$). If we find that moving $p$ upwards (in the positive direction of $\tilde{S}$) causes $f(p)$ or $f(\tilde{S})$ to rotate in the positive direction (counterclockwise) around $p$, we know (by Exercise 16, Chapter 5) that our guess at the direction of $\mathcal{K}$ is correct.

Suppose that, as with $\log(z)$, $S$ is a segment that passes through the origin and $p$ when $\mathcal{K}$ vanishes at $p$. If $p = re^{i\theta}$, then the direction of $S$ is $\theta$. $\mathcal{K}$ is necessarily orthogonal to $S$, but it may point in either of two directions, $\pm i\hat{\xi}$. We guess that the direction of $\mathcal{K}$ is $i\hat{\xi}$, where $\hat{\xi}$ is a unit vector on $S$ at $p$. We rotate $S$ around $p$ in either direction until $S$ parallels $\mathcal{K}$, and $S$ and $\mathcal{K}$ point in the same direction (as seen by the simultaneous rotation of $\hat{\xi}$). Now we move $p$ along $\tilde{S}$ in the direction chosen for $\mathcal{K}$. This case is different, because the angle of $p$ now changes as $p$ moves. If we find that $f(p)$ moves in the positive direction (counterclockwise), then we would know we had the correct direction of $\mathcal{K}$. If we find that $f(p)$ rotates in the negative direction around $p$, we know that $\mathcal{K}$ should point in the direction of $-i\hat{\xi}$.

Apply to $\log(z)$

From (i), we decided that in $f(S) = e^x e^{iy}$, $y$ must be a constant, so $f(S)$ is parallel to the real line (Figure 2). The complex curvature could point in the direction of $\pm i$. We know from exercise 16, chapter 5, that when curvature is positive, an infinitesimal tangent to $f(z)$ points in the direction of positive rotation. If we rotate $S$ to $\tilde{S}$ and move $p$ in the direction of increasing $y$, $f(\tilde{S})$ rotates in the positive direction inscribing a circle of radius $e^x$ (Figure 2). $\mathcal{K}$ must therefore point in the direction of $+i$. The curvature $\kappa$ is positive with magnitude $e^{-x}$. Then $\kappa$ is at its maximum absolute value and $|\mathcal{K}| = |\kappa| = e^{-x}$. Since $\mathcal{K}$ points in the direction of $i$, $\mathcal{K} = i|\mathcal{K}| = ie^{-x}$.

(iii) Use (28) to verify this formula.

(28) \[ \mathcal{K} = \frac{if'}{f'|f'|} \]
\[(e^x)'' = (e^x)' = e^x\]

\[K = \frac{i \overrightarrow{e' \cdot |e'|}}{e'^2} = \frac{i}{|e'|} = ie^{-x}\]

END OF (iii)

Figure 3. Log(z)
on arbitrary directed line segment S

Figure 4a. Log(z) on S, \(K \hat{\xi} = 0\)
p moves in direction of \(i \hat{\xi}\)
(iv) Repeat as much as possible of the above analysis in the cases \( f(z) = \log(z) \) ...

In neither of these cases will you be able to see the exact value of \( |\mathcal{K}(p)| \).

Remembering that \( f(S) \) must be a line segment, we rewrite \( \log(p) \) as

\[
\log(p) = \ln r + i \theta, \quad \text{where} \quad r = |p| \quad \text{and} \quad \theta = \arg(p)
\]

We can see from the equation that \( \log(z) \) is a line segment only if \( \theta \) is constant. Since \( p = r e^{i\theta} \) is a point in \( S \), \( S \) must be coincident with \( p \), and \( \log(z) \) must be a line parallel the real line. The complex curvature points in the direction \( \pm i\hat{\xi} \). Which?

We plot \( S \) (Figure 4a) and we guess that the direction of \( \mathcal{K} \) is \( i\hat{\xi} \), which is the direction of positive rotation of \( S \) (and opposite to the true direction of \( \mathcal{K} \) shown in Figure 4a). We rotate \( S \) around the origin to be parallel with our hypothetical \( \mathcal{K} \) by multiplying all points in \( S \) by \( i \). Now we examine \( \log(p) \) on a \( p \) that moves parallel to \( \mathcal{K} \).

\[
\log(p) = \ln r + i \arctan(y/x)
\]

Now, as \( p \) moves, its direction changes.

We plot \( f(S) \) by moving \( p \) in the direction of \( i\hat{\xi} \) as shown by the dotted arrow \( S \) in Figure 4a. We see that the \( f(S) \) is plotted as an arrow curving to the right around \( p \). If we move \( p \) in the opposite direction towards \(-i\hat{\xi} \) (Figure 4b), \( f(S) \) curves in the positive direction. The direction of \( \mathcal{K} \) is in fact shown by \(-i\hat{\xi} = \)
$-e^{j\theta}$ as in Figure 4b.

We can not see an exact value of $\tilde{\kappa}$ at $p$ by inspection of the plot because we can not determine $|\log(p)|$.

Use (28) to find $\mathcal{K}$:

Let $\theta = \arg(p)$.

For $\log(z)$,

$$(\log(z))' = z^{-1}$$

$$(\log(z))'' = -z^{-2}$$

Then

$$\frac{f^n}{f'} = \frac{-z^{-2}}{z^{-1}} = -z^{-1} = -\frac{e^{j\theta}}{r} = \frac{\partial}{r}$$

$$\mathcal{K} = i \left( \frac{\partial}{r} \right) \frac{1}{|z^{-1}|} = i \left( \frac{\partial}{r} \right) r = -ie^{j\theta}$$

This confirms our conclusion from inspection of the curve of $f(S)$ that $\mathcal{K}$ points in the direction of $-ie^{j\theta}$.
(iv) Repeat as much as possible of the above analysis in the cases ... $f(z) = z^m$, where $m$ is a positive integer.

We will illustrate this with a plot of $z^2$. We rewrite $z^m = r^m e^{im\theta}$. Then, for a given $m$, $z^m$ is a straight line segment only if $\theta$ is constant and $S$ is coincident with $p$. With vanishing curvature at $p$, $\mathcal{K}$ is orthogonal to $S$, so $\mathcal{K}$ must point in either of two directions, $\pm i\xi$. We again rotate $S$ in the direction of $i\xi$ and plot $f(S)$. As $p$ moves along $S$ in the direction of $i\xi$, $f(p)$ curves in the positive direction (Figure 5). We conclude that the direction of $\mathcal{K}$ is the same as $ie^{i\theta}$. Inspection of the plot does not seem to provide an
exact formula for \( \mathcal{K} \) except at \( f(p) \), where the radius of a circle of curvature is \( |p|^m \). At that point only, \( \tilde{\kappa} = 1/|p|^m \).

From (28),

\[
\mathcal{K} = \frac{i \cdot m \cdot (m-1) \cdot z^{m-2}}{|z_m^{m-1}|^m} = \frac{i \cdot (m-1) \cdot \left( \frac{z^{m-2}}{z^{m-1}} \right)}{m^{m-1}} = \frac{i \cdot (m-1)}{m^{m-1}} \cdot z^{-1} = \frac{i \cdot (m-1)}{m^{m-1}} \cdot e^{i \theta}
\]

In the case of \( z^2 \), \( \mathcal{K} = (i/2) \cdot e^{i \theta} \).

(v) According to the geometric reasoning in Ex. 18, p. 213, the amplification of a Möbius transformation \( M(z) = \frac{az+b}{cz+d} \) is constant on each circle centred at \(-d/c\). Thus the complex curvature of \( M \) should be tangent to these concentric circles. Verify this by calculating \( \mathcal{K} \).

A circle centered at \(-d/c\) may be described as \( |z-d/c| = \text{const.} \)

\[
M(z-c/d) = a(z-d/c) + b = \frac{az-ad/c + b}{cz-d + d} = \frac{az+bc/cz-ad/c^2}{cz} = \frac{a/c + b/cz - ad/c^2}{cz}
\]

\[
= \frac{1}{cz} (b/c - ad/c) + a/c
\]

\[
= \frac{1}{cz} (bc/c^2 - ad/c^2) + a/c
\]

\[
= \frac{1}{cz} (-ad/bc/c^2) + a/c
\]

\[
= \frac{1}{cz} (-1/c^2) + a/c \quad \text{[ad-bc \( \rightarrow 1 \) under normalization]}
\]

Equation (1) describes the steps of the decomposition of the Möbius transformation (p. 124).

The amplification is \( |M'| \).

\[
M' = \left[ \frac{1}{cz} (-1/c^2) + a/c \right]' = \frac{-z^{-2}}{c^2} = \frac{1}{c^2 \cdot z^2}
\]

\[
|M'| = \left| \frac{1}{c^2 \cdot z^2} \right|
\]

\[
= \frac{1}{c^2 \cdot r^2} = \text{const.}
\]

\[
M'' = \left( \frac{1}{c^2 \cdot z^2} \right)' = \frac{-2z^3}{c^2}
\]
\[ \mathcal{K} = \frac{i(-2z^{-1})}{c} = i\left(\frac{-2z^{-1}}{c^2}\right) c^2 z^2 c^2 r^2 \]
\[ = ic^2 r^2 (-2 z^{-1}) \]
\[ = -\frac{2c^2 r^2}{z} \]
\[ = -\frac{2c^2 r^2}{re^{i\theta}} = -2c^2 r e^{i\theta} = -2c^2 iz \]

We see that \( \mathcal{K} \) points in the direction \(-iz\), which is tangent to the circle \(|z-d/c| = \text{const.}\).

(vi) Use a computer to verify figure [21] for all four mappings above.

For (vi) we are going to need \( \mathcal{R} \) and \( \mathcal{E} \) (p. 240-241), \( \sigma \) (p. 237), and \( \alpha \), which Vasco defined as the extra amplification. The derivation of \( \alpha \) is analogous to that of \( \sigma \). Refer to [18] for \( \nu \) (nu).

\[ \alpha = \text{Re}(\nu) = \text{Re}\left(\frac{f}{r}\right) = \text{Re}\left(\frac{r^* \xi}{r r^*}\right) = |f||\xi| \text{Re}\left(\frac{r^* \xi}{|r|}\right) = |f||\xi| \text{Re}\left(i\overline{\mathcal{K}} \hat{\xi}\right) \]
\[ = |f||\xi| \text{Re}\left(i\left(\mathcal{K} \hat{\xi} + \mathcal{E} \times \hat{\xi}\right)\right) = |f||\xi| \text{Re}\left(-\mathcal{K} \times \hat{\xi} + \mathcal{E} \hat{\xi}\right) = -|f||\xi| \mathcal{K} \times \hat{\xi} \]
\[ = |f||\xi| \mathcal{E} \]

\[ \sigma = \text{Im}(\nu) = \text{Im}\left(\frac{\hat{\xi}}{r}\right) = \text{Im}\left(\frac{r^* \xi}{r r^*}\right) = |f||\xi| \text{Im}\left(\frac{r^* \xi}{|r|}\right) = |f||\xi| \text{Im}\left(i\overline{\mathcal{K}} \hat{\xi}\right) \]
\[ = |f||\xi| \text{Im}\left(i\left(\mathcal{K} \hat{\xi} + \mathcal{E} \times \hat{\xi}\right)\right) = |f||\xi| \text{Im}\left(-\mathcal{K} \times \hat{\xi} + \mathcal{E} \hat{\xi}\right) = |f||\xi| \mathcal{K} \hat{\xi} \]
\[ = |f||\xi| \mathcal{R} \]
Figure 6. Circles and bars under $e^z$
Cf. Needham Figure [21]

$e^z$

Apparently speaking generally in (31), Needham says $\tilde{Q}$ rotates most rapidly and its size remains constant, when $Q$ moves in the direction of $\mathcal{K}$ and $\tilde{Q}$ expands most rapidly, and does not rotate, when $Q$ moves in the orthogonal direction -$\mathcal{K}$ (p. 240). The RHS of Figure 6 appears to validate Needham’s description. The greatest rotation (positive or negative) occurs on the white circles which lie on the same $\xi$ lines as $\mathcal{K}$ and -$\mathcal{K}$. The greatest expansion is visible RHS circle which corresponds to the circle at -$\mathcal{K}$ = 1 on the LHS.

The solution to this problem can be found in Figure [18]. We let $c = p$ and $c’ = q$. The distance from $c$ to $c’$ corresponds to $\xi$. The idea is to see the effect of amplitwist on $\xi$ and $\zeta$ and then add a bit of extra amplitwist. The angles of nPi/3 between the $\xi_n$ might tempt one to add nPi/3 to the twist of the amplitwist operating on $\zeta$, but that would be a mistake. Figure [18] shows that the twist of $\zeta$ consists entirely of the amplitwist at $p$ plus a bit of extra twist labeled $\sigma$. Needham does not add in the size of the angle between $p$ and $q$. Additionally, we have to consider a bit of extra amplification, which Vasco has labeled $\alpha$.

Let $c = i$ be the center of $Q$ (A choice on the real line would be uninteresting.)

$$f(c) = f'(c) = f''(c) = e^i.$$ 

Let $\xi_n = e^{n\pi i / 6}$, $n = 1..12$. Then the centers of the satellite $Q’$s are $c’_n = c + \xi_n$. Let the center $\tilde{c}$ of $\tilde{Q}$ equal $f(c) = e^i$. Apply amplitwist to $\xi_n$. The amplitwist is $f’(c) = e^i$. This is a rotation by 1 radian.
\( \tilde{\xi}_n = f(c) \xi_n = e^i \xi_n = \{ e^{i\frac{\pi}{6}}, e^{i\frac{\pi}{3}}, e^{i\frac{\pi}{2}}, \ldots \} = \{ e^{i(1+\frac{\pi}{6})}, e^{i(1+\frac{\pi}{3})}, \ldots \} \)

We find the centers of the satellite circles \( \tilde{Q}_n \) on the RHS.

\( (c')_n = \tilde{\xi}_n + \tilde{c} \)

To plot the circles \( \tilde{Q}_n \) on the RHS, we need to amplify the radius, which we can think of as an infinitesimal number. We don’t need to twist the \( \tilde{Q}_n \), because we couldn’t see the twist anyway. The twist will be shown by the bar/lever. We need to give the radius extra amplification \( \alpha \) to account for its distance \( |\xi| \) from \( f(c) \).

The radius \( r \) with amplification and extra amplification is:

\[ \tilde{r}_n = \alpha |f'(c)| r \]

where \( \alpha = 1 + |\xi_n| \sin \gamma = 1 + \sin(\frac{\pi}{2} - \phi) = 1 + \sin(\frac{\pi}{2} - \text{Arg}(\xi_n)). \) Since we drew the original \( Q_n \) on the unit circle, \( |\xi_n| = 1 \) for all \( n \).

Hence,

\[ \tilde{r}_n = (1 + \sin(\frac{\pi}{2} - \text{Arg}(\xi_n))) |e^i| r = (1 + \sin(\frac{\pi}{2} - \text{Arg}(\xi_n))) r \]

Notice that total amplification is equivalent to \( \alpha \) because \( |f'(c)| = 1 \). We can use this fact in calculating \( \tilde{\xi}_n \).

\[ \text{totalAmplification} = (1 + \sin(\frac{\pi}{2} - \text{Arg}(\xi_n))) = \alpha \]

For derivations, see Vasco’s document, where \( \phi = \text{Arg}(\xi_n) \).

Plot the circles \( \tilde{Q}_n \) with these radii at centers \( (c')_n \).

Now it only remains to plot the bars/levers, which are amplitwisted versions of the little identical horizontal infinitesimal \( \tilde{\xi}_n \) on the LHS. Apply amplitwist and extra amplitwist. We already know the total amplification. We need the total twist \( \omega \) of \( \tilde{\xi}_n \). This will be the twist plus the extra rotation \( \sigma \).

\[ \sigma = \cos(\frac{\pi}{2} - \text{Arg}(\xi_n)) \]

The twist part of \( f'(c) = e^i \), which has a twist angle of 1 radian. We do not add the angles of \( \xi_n \). Just the twist. Following the analogy with Figure [18], we use \( \epsilon \) for the resulting angle.

\[ \epsilon = \text{Arg}(f'(c)) + \sigma = \text{Arg}(e^i) + \sigma = 1 + \sigma \]

Remembering that the rotation of \( \xi \) is zero, we write the template and then expand it.
\[\tilde{\zeta}_n = \alpha |\zeta| e^{i\epsilon} = \alpha |\zeta| e^{i(1 + \sigma)}\]

\[= (1 + \sin(\pi/2 - \text{Arg}(\tilde{\zeta}_n)))|\zeta| e^{i(1 + \cos(\pi/2 - \text{Arg}(\tilde{\zeta}_n)))}, \text{ by substitution}\]

Plot \(\tilde{\zeta}_n\) at \((\tilde{c})_n\).

\[\mathrm{Log}(z)\]

Figure 7. Circles and bars under \(\mathrm{Log}(z)\)

Cf. Needham Figure [21]

17 (vi) \(\mathrm{Log}(z)\)

From above, the complex curvature of \(f(z) = \mathrm{Log}(z)\) is \(\kappa_{\mathrm{Log}(z)} = -ie^{i\theta}\), showing that, unlike the case with \(f = e^z\), \(\kappa\) depends on the argument of \(z\), because \(z\) is the point at which we apply \(\mathrm{Log}(z)\). In this case it is \(c\), the center of \(Q\) on the LHS, so we could write \(\kappa(f(c))\) showing that \(\kappa\) is intrinsic to \(f\) at a particular point, but generally.

Place the center of \(Q\) at \(c \neq 0\), because \((\log(z))' = 1/z\), which is undefined at \(z = 0\). If \(Q\) were centered at the origin, the perimeter of \(Q\) would plot as a vertical line under \(\mathrm{Log}(z)\). Let \(r\) be the radius of \(Q\) and \(q\) be the center. If we offset \(q\) by 2 \(r\) or more, the result approximates a circle on the RHS. Translate \(Q\) outwards away from its center along 12 different \(\xi_n\), as shown in Figure 7(a). Plot \(\tilde{Q}\) as a circle centered at \(\mathrm{Log}(c)\) with radius \(r\) amplified by \(|f'(q)| = |1/c|\). Then find the peripheral circles on the RHS centered at \(c = (f(c)\xi_n + \mathrm{Log}(c))\) with radii of \(\alpha(\xi_n)|f'(c)| r\) and plot new circles on the RHS. Amplitwist the bars with \(f'(c)\zeta\) and apply extra amplification and twist, as described below.
As in the previous, let $c$ be the center of $Q$, let $r$ be the radius of $Q$, let $\xi_n$ be numbers representing the movements of $Q$ away from $c$, let $\zeta$ be a number representing the length and direction of the horizontal bars, and let a tilda indicate a transformation from (a) to (b). This could be $f(c)$, an amplification, a twist, or some combination of these.

We have used the numbers $r = .1, c = 1.05 e^{i\pi/3}, |\xi| = .75, |\zeta| = 1.5 r$.

For the Log(z) function

$$|\mathcal{K}| = | -ie^{i\theta} | = 1$$

$\mathcal{E}$ and $\mathcal{R}$ are related to $\xi$, $\alpha$, $\sigma$, and $\mathcal{K}$ as follows:

$$\mathcal{R} = \text{Im} \left[ \frac{r'(p) \xi}{r'(p) |f'(p)| |\xi|} \right] = \hat{\xi} \times \mathcal{K} = \cos(\text{Arg}(\mathcal{K}) - \text{Arg}(\hat{\xi})) = \cos(\text{arg}(\mathcal{K}/\hat{\xi}))$$

$$\mathcal{E} = \text{Re} \left[ \frac{r'(p) \xi}{r'(p) |f'(p)| |\xi|} \right] = \hat{\xi} \times \mathcal{K} = \sin(\text{arg}(\mathcal{K}/\hat{\xi}))$$

$$\sigma(\xi) = \text{Im} \left[ \frac{f'(c) \xi}{f'(c)} \right] = f'(p) |\xi| \mathcal{R} = |f'(p)| |\xi| \cos(\text{arg}(\mathcal{K}/\hat{\xi}))$$

$$= \left| \frac{\xi}{c} \right| \cos(\text{arg}(\mathcal{K}/\hat{\xi})) \quad \text{[the extra twist]}$$

$$\alpha(\xi) = 1 + \text{Re} \left[ \frac{r'(p) \xi}{r'(p) |f'(p)| |\xi|} \right] = 1 + |f'(p)| |\xi| \sin(\text{arg}(\mathcal{K}/\hat{\xi}))$$

$$= 1 + \left| \frac{\xi}{c} \right| \sin(\text{arg}(\mathcal{K}/\hat{\xi})) \quad \text{[the extra amplification]}$$

Calculate the amplification of the radius of $Q$.

$$\tilde{r} = \alpha(\xi) |f'(c)| r \quad (r \text{ is the radius of } Q, \text{ not } |c|)$$

$$= \alpha(\xi) \frac{r}{|c|}$$

Calculate the amplitwist applied to the movements $\xi_n$.

$$\tilde{\xi}_n = |f'(c)| \xi_n = \frac{\xi_n}{|c|}$$

Use $\alpha$ and $\sigma$ to calculate $\tilde{\zeta}_n$.

$$\tilde{\zeta}_n = \alpha(\xi_n)|f'(c)||\xi| e^{i(\text{Arg}(f'(c)) + \sigma(\xi_n))}$$
\[
\zeta = \alpha(\xi_n) \left| \frac{\xi}{c} \right| e^{i(\text{Arg}(1/c) + \phi(\xi_n))}
\]

The dashed circles are unit circles.

The script that plotted Figure 7 represents these equations precisely. Plot (b) appears to conform to Needham’s figure [21] and to Vasco’s plot. The rotated version of (b) on the lower right returns the circles to their orientation prior to amplitwisting so that the rotation of the levers can be seen more clearly.