15. (i) By noting the unit tangent (in the counterclockwise direction) to an origin centred circle can be written as $\xi = i(z/|z|)$, show that formula (23) for the curvature of the image of such a circle can be written as

$$\kappa = \frac{1 + \Re \left( \frac{zf''}{f'} \right)}{|zf'|}$$

Substitute $i(z/|z|)$ for $\xi$.
Substitute $\frac{1}{z}$ for $k$, because the equation for the circle can be written $z = r e^{i\theta}$.

Let $f$ be understood to be $f(z)$.

$$\kappa = \frac{1}{|f'(p)|} \left( \Im \left( \frac{f''(p) \xi}{f'(p)} \right) + \kappa \right) \quad (23)$$

(ii) What should this formula yield if $f(z) = \log z$? Check that it does.

$\log z$ sends circles to vertical lines. Since the circle is origin-centred, $\log z$ should send the circle to the $y$ axis. Then $\kappa = 0$.

$$f' = (\log z)' = \frac{1}{z}$$

$$f'' = -z$$

$$\kappa = \frac{1 + \Re \left( \frac{zf''}{f'} \right)}{|zf'|} = \frac{1 + \Re \left( \frac{f''(p) \xi}{f'(p)} \right)}{|zf'|} = \frac{1 + \Re \left( \frac{-z}{f'(p)} \right)}{|zf'|} = 0$$
(iii) What should this formula yield if \( f(z) = z^m \)? Check that it does. What is the significance of the value of \( \kappa \) when \( m \) is negative? [Hint: Which way does the velocity complex number of the image rotate as \( z \) travels counterclockwise around the original circle?]

The absolute value of \( z \)

\[
|z|
\]

\[
f'(z) = (z^m)' = m z
\]

\[
f''(z) = m(m - 1)
\]

\[
\kappa = \frac{1 + \text{Re} \left[ \frac{z(m-1)z^{m-1}}{mz^{m-1}} \right]}{|zmz^{m-1}|}
\]

\[
= \frac{1 + \text{Re} \left[ \frac{(m-1)z^{m-1}}{z^{m-1}} \right]}{|mz^m|}
\]

\[
= \frac{1 + \text{Re}(m-1)}{|mz^m|} = \frac{1 + m - 1}{|mz^m|} = \frac{m}{|m| |z|^m}
\]

\[
= \text{sign}(m)
\]

\[
|z|
\]

The formula results in

\[
|z|
\]

What is the significance of the value of \( \kappa \) when \( m \) is negative? [Hint: Which way does the velocity complex number of the image rotate as \( z \) travels counterclockwise around the original circle?]

We consulted chapter 1, §II, 2 “Moving Particle Argument” (pp. 10-12) for the following. Let \( z = \)
We see that when $m > 0$, $V$ rotates $mw$ by $\pi/2$ in the counterclockwise direction, so it is tangent to the image circle. As $z$ travels in the counterclockwise direction, $V$ must also rotate in the counterclockwise direction (Figure 1).

To illustrate the case when $m < 0$, let $n = -m$, i.e. $n$ is a positive number, so $V = imr$.

Figure 1 (a) shows a circle of radius .8 and a gray, dashed ray pointing to .8. This same arrow under $f$ is shown in (b). Under $f(z) = zr$. 

$r$
\( f(z) = z^5 \)