12. Provided it is properly interpreted, show that $(z^\mu)' = \mu z^{\mu - 1}$ is still true even if $\mu$ is complex.

The qualifying phrase “provided it is properly interpreted” suggests that $z^\mu$ might be a multifunction with many values. We can rewrite $z^\mu = e^{\mu \log z}$. We let $\mu = a + ib$. If $b = 0$, then $\mu = a$ and $z^\mu = e^{a \log z}$. We examine the possibilities in case $b = 0$.

If $\mu$ is a real number or integer

$$(z^a)' = (e^{a \log z})'$$

$$= e^{a \log z} (a \log z)'$$

$$= e^{a \log z} \left( \frac{a}{z} \right)$$

$$= a \frac{z^a}{z}$$

$$= a z^{a-1}$$

This satisfies the equation $(z^\mu)' = \mu z^{\mu - 1}$.

If $a = p/q$ is a rational fraction, then $z^\mu$ is a multifunction with many values (pp. 101-102, 230), as seen by the first term of the following result, which contains the integer $n$ in the numerator of the power. Square brackets signify the principal branch of the log function, as does the capital L on Log. It is given, where $z = r e^{i \theta}$, that $\text{Log } z = \ln r + i \theta$ and $\log z = \text{Log } z + i2n\pi$.

$$z^{p/q} = e^{(p/q) \log z}$$

$$= e^{(p/q) (\text{Log } z + i2n\pi)}$$

$$= e^{(p/q) \text{Log } z} e^{i2n\pi}$$

$$= e^{\frac{i2n\pi}{3}} [z^{p/q}]$$

(1)

Then $(z^\mu)' = (e^{\frac{i2n\pi}{3}} [z^{p/q}])'$

$$= e^{\frac{i2n\pi}{3}} [z^{(p/q - 1)}]$$

$$= e^{\frac{i2n\pi}{3}} \left( \frac{(p/q) [z^{p/q}]}{z} \right)$$

(2)
Needham (p. 230) says, where “a” is a fraction,

The amplitwist of each branch of \( z^a \) is given by \( (z^a)' = a z^a / z \), provided that the same branch of \( z^a \) is used on both sides of the equation. We substitute (1) into the LHS and (2) into the RHS:

\[
(e^{\frac{\ln(p/q)}{3}} [z^{p/q}])' = e^{\frac{\ln(p/q)}{3}} \frac{(p/q) [z^{p/q}]}{z}
\]

\[
e^{-\frac{\ln(p/q)}{3}} [z^{p/q}] = e^{\frac{\ln(p/q)}{3}} \frac{(p/q) [z^{p/q}]}{z}, \text{ by dividing both sides by the branch factor } e^{\frac{\ln(p/q)}{3}}
\]

Substituting a for \( p/q \) and \( z^a \) for \( [z^{p/q}] \), gives \( (a z^a) / z \), so our calculation bears out the equation for amplitwist provided that the same branch of \( z^a \) is used on both sides of the equation. The equation for amplitwist of a fractional exponent of a complex number satisfies the equation \( (z^a)' = \mu z^{a-1} \).

In the case that \( \mu \) is complex, let \( \mu = a + ib \). Then

\[
z^a = e^{\mu \log z}
\]

\[
= e^{\mu (\ln r + i(\theta + 2n\pi))}
\]

\[
= e^{i2n\pi} e^{\mu \log z}
\]

\[
= e^{i2n\pi} [z^a]
\]

\[
(z^a)' = (e^{i2n\pi} e^{\mu \log z})'
\]

\[
= e^{i2n\pi} (e^{\mu \log z} (\mu \log z)')
\]

\[
= e^{i2n\pi} (\mu [z^a]) (\log z)'
\]

\[
= e^{i2n\pi} (\mu [z^a]) (1/z)
\]

Then

\[
(e^{i2n\pi} [z^a])' = e^{i2n\pi} \mu [z^a]/z
\]

Divide by the branch term on both sides

\[
[z^a]' = \mu [z^a]/z = \mu [z^{a-1}]
\]
Again, provided it is properly interpreted so that the same branch factors are used on both sides of the equation, we can write \((z^\mu)' = \mu z^{\mu-1}\).

As a point of interest, the branch factor \(e^{i\mu 2n\pi}\) expands to \(e^{-2n\pi(b-ia)}\).