Consider the mapping \( f(z) = z^4 \) illustrated above. On the left is a particle \( p \) travelling upwards along a segment of the line \( x = 1 \), while the right is the image path traced by \( f(p) \).

(i) Copy this diagram, and by considering the length and angle of \( p \) as it continues on its upward journey, sketch the continuation of the image path. See Figure 1.

As \( p \) continues upward, the angle \( \theta \) between the ray to \( p \) and the y axis increases and approaches \( \pi/2 \) as \( \text{Im}[p] \) goes to infinity. If we write \( z^4 = r^4 e^{i4\theta} \), we see that the image of \( p \) makes one revolution around the origin between \( \theta = 0 \) and \( \pi/2 \), because \( 4(\pi/2) = 2\pi \).

Then if we write \( z^4 = (1 + iy)^4 \), we see that as \( \text{Im}[p] \) goes to \( \infty \), \( \text{Re}[\tilde{p}] \) goes to \( \infty \) and \( \text{Im}[\tilde{p}] \) goes to \( 4\infty^3 \).

\[
\lim_{\text{Im}[p] \to \infty} (z^4) = (1 + i\infty)^4 = (1 + 2i\infty - \infty^2)^2 \rightarrow \infty^4 - 4\infty^3 i
\]

By writing \( z^4 = u + iv = (x^4 - 6x^2y^2 + y^4) + i(4x^3y - 4xy^3) \), we also see that the path of \( p \) crosses the y axis when \( (x^4 - 6x^2y^2 + y^4) = 0 \), so \( 1 - 6y^2 + y^4 = 0 \), and

\[
y^2 = \frac{6 \pm \sqrt{36 - 4}}{2} = \frac{6 \pm \sqrt{32}}{2} = \frac{6 \pm 4\sqrt{2}}{2} = 3 \pm 2\sqrt{2}
\]

Then \( y = \pm\sqrt{3 \pm 2\sqrt{2}} \).

This would lead to four values of \( z^4 \). Since we are interested only in the upward path of \( p \) and the image path \( \tilde{p} \) beginning at 1 and looping up and then downward (curving positively, counterclockwise) in Figure 1, we choose \( y = \sqrt{3 - 2\sqrt{2}} \), which is \( \text{Im}[p] \) when \( \theta = \pi/8 \) and \( \tilde{p} = 1.37258i \) (at A), and we choose \( y = \sqrt{3 + 2\sqrt{2}} \), which is \( \text{Im}[p] \) when \( \theta = 3\pi/8 \) and \( \tilde{p} = -46.6274i \) (See red dots in Figure 1).

Figure 1 shows the path of the image of \( p \) curving in a positive direction down with acceleration towards the origin, but the path goes to powers of infinity on both the x and y axes. If we were to take the path of \( p \) from \( \text{Im}[p] = -\infty \) to \( \infty \), or \( \theta = -\pi/2 \) to \( \pi/2 \), then we could see that it is possible for the image of \( p \) to make two full loops around the origin. As it is, we see only one partial loop. In Figure 2 b, we see two half loops.

(ii) Show that \( A = i \sec^4(\pi/8) \).

The following are true at any point \( z = x + iy \). See Figures 2a and 2b.

\[
x = r \cos \theta, \quad y = r \sin \theta
\]
\[ r^2 = 1^2 - r^2 \sin^2 \theta \]
\[ r^2 + r^2 \sin^2 \theta = 1 \]
\[ r^2 \left(1 + \sin^2 \theta\right) = 1 \]
\[ r^2 = \frac{1}{1 + \sin^2 \theta} \]

\[ r = \pm \frac{1}{\sqrt{1 - \sin^2 \theta}} = \sec \theta \quad \text{[trig identity]} \quad (1) \]

Let \( z = re^{i\theta} \). Then
by taking the 4th power
\[ A = z^4 = r^4 e^{4i\theta} \quad (2) \]
by substitution
\[ A = \sec^4(\theta) e^{i\text{arg}(A)} \quad (3) \]
by construction, \( \text{arg}(A) = \pi/2 \)
\[ A = \sec^4(\theta) e^{i(\pi/2)} \quad (4) \]
Set \( \text{arg}(z^4) \) in (2) equal to \( \text{arg}(\sec^4(\theta) e^{i(\pi/2)}) \) in (4).
\[ 4\theta = \pi/2 \quad \Rightarrow \quad \theta = \pi/8 \]

Since \( \text{arg}(A) = \pi/2 \) and \( e^{i(\pi/2)} = i \), \( A = i \sec^4(\pi/8) \).

Figure 1. Continuation of \( z^4 \) on \( x = 1 \)
from \( \text{Im}[p] < \sqrt{3 - 2 \sqrt{2}} \) to \( \text{Im}[p] > \sqrt{3 + 2 \sqrt{2}} \).
(iii) Find and mark on your picture the two positions (call them $b_1$ and $b_2$) of $p$ that map to the self-intersection point $B$ of the image path.

$z^4 = u$ because $vi = 0$

Let $z = x + iy$.

$$u = \text{Re}[(x + iy)^4] = x^4 - 6x^2y^2 + y^4$$

$$v = \text{Im}[(x + iy)^4] = 4x^3y - 4xy^3 = 0 \implies y = \pm 1, \text{ because } x = 1$$

The two points must be $1\mp i$ (Figure 2a).
\[ b_1 = 1 + i \]
\[ b_2 = 1 - i \]
\[ B = (1 \pm i)^4 = -4 \]

This value can also be obtained by substituting ±1 into the formula for u.

(iv) Assuming the result \( f'(z) = 4 \, z^3 \), find the twist at \( b_1 \) and also at \( b_2 \).

We can see from Figure 2a, that \( \theta \) at \( b_1 \) is \( \pi/4 \) (or we could calculate it from \( \arctan(1) = \pi/4 \)).

The twist is \( \arg(f(z)) = \arg(4 \mid z \mid e^{i3\theta}) = 3\theta = \pm3\pi/4 \).

(v) Using the previous part, show that (as indicated at B) the image path cuts itself at right angles.

Since \( z^4 \) is analytic and conformal, the twist and the argument of the tangent are the same: \( \arg(z^4) \) at \( z = (1 \pm i) \). The tangents in this case have the same angles as the twist, \( \pm3\pi/4 \). Angle \(-3\pi/4\) is equivalent to \( 5\pi/4 \). Then

\[ 5\pi/4 - 3\pi/4 = \pi/2, \text{ which is a right angle.} \]

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Other observations not bearing directly on the questions.

Figure 3 plots \( \epsilon \) vectors emanating from B at tangent angles. Multiplying both \( \epsilon \) tangents by the amplitwist illustrates that \( z^4 \) is conformal.
Figure 3. Amplitwist of $z^4$ on $x = 1$