18. Recall Ex. 19, p. 186, where you showed that a general Möbius transformation

\[
M(z) = \frac{az + b}{cz + d}
\]

maps concentric circles to concentric circles if and only if the original circle (call it \( \mathcal{C} \)) is centred at \( q = -d/c \). Let \( \rho = |z-q| \) be the distance from \( q \) to \( z \), so that members of \( \mathcal{C} \) are \( \rho \equiv \text{const.} \)

```mathematica
Clear["Global`"]

(* PROGRAM FOR FIGURE 1. CONCENTRIC CIRCLES UNDER M(Z). Added November 15, 2015, revised November 21, 2015. *)

(* CONSTANTS. Keep a,b,c,d within narrow range to avoid imploding or exploding. *)

(* In this version, \( \rho = |w| \), and \( z = w + q \), where \( q = -d/c \). *)

pr = 2;

(* a, b, c, d, and abse are all arbitrarily chosen, but it makes the plot more manageable if a, b, c, and d have similar absolute values. *)

a = Exp[I Pi / 3];
b = Exp[I 1.9 Pi / 3];
c = Exp[I 4 Pi / 3];
d = Exp[I 5 Pi / 4];
q = -d / c;

abse = .3; (* All the e vectors in \( \mathcal{C} \) are same size and direction (blue arrows). *)

cNumbs = {0, 1, a, b, c, d};
nCircles = 15;
firstCirc = 3; (* first circle displayed *)
lastCirc = 8; (* last circle displayed *)

(* FUNCTIONS *)

v[z_] := {Re @ z, Im @ z}; (* Complex to vector *)
m[z_] := (a z + b) / (c z + d); (* the Möbius transformation *)
```
SetAttributes[{v, m}, Listable]; (* enables v and m to process lists *)

(* amplification = \|M'(w)\| \|w\| = \rho *)
(* For this problem, twist is applied to points in circles centered at origin.
For convenience, the points are usually calculated as ne, n a positive integer,
e a small vector. w = z-q = ne, n a pos. integer indexing a circle in \mathcal{F},
e a small vector orthogonal to a circle in \mathcal{F}. *)

ampli[w_] := Abs[(a d - b c)] / (Abs[c]^2 Abs[w]^2);

(* twist = Arg[M'(z)]; z is point on circle centered at q *)
(* w = z-q is an origin-centered point from which e emanates. *)

twist[w_] := Arg[(a d - b c) / ((c)^2 w^2)];

(* w is point from which e emanates; w = z-q *)

amplitwist[w_] := ampli[w] Exp[I twist[w]];

(* GRAPHICS OBJECTS *)

(* POINTS *)

points = Point @ v @ cNumbs;
qPt = {PointSize @ .01, Point @ v @ q};
acPt = {Gray, PointSize @ .01, Point @ v[a/c]};
points = {points, acPt, qPt};

(* LINES *)
xAxis = {Gray, Line @ v @ {-2 pr, 2 pr}};
yAxis = {Gray, Line @ v @ {-pr I, 2 pr I}};
size = .01;

\thetaArrows = 3 Pi / 4; (* arbitrary angle for arrows on \mathcal{F} *)
\epsilon = abse Exp[I \thetaArrows]; (* a small complex number for a vector *)
eArrows = Table[ {Blue, Arrowheads @ size, Arrow[v @ {n e + q, (n + 1) e + q}]},
{n, Range[1, nCircles]}];

(* amplitwisted \epsilon arrows at w + q, where w = n\epsilon *)
ateArrows = Table[ {Green, Arrowheads @ size,

        Arrow[v @

              m [n \epsilon + q], (* point of emanation of vector under M(z) *)}]


m[n e + q] + amplitwist[n e] e (* w = ne *)
}

{n, Range[1, nCircles - 1]];

lines = {xAxis, yAxis,
eArrows[[firstCirc ;; lastCirc]], atArrows[[firstCirc ;; lastCirc]]};

(* ARCS *)

(* points for circles in F *)
theFPts =
  Table[Table[(n Abs @ e) Exp[I θ] + q, {θ, 0, 2.1 Pi, .01}], {n, Range[1, nCircles]}];

(* points for circles in M(F) *)
mFPts = Table[Table[m @ pt, {pt, circle}], {circle, theFPts}];

(* F *)
zCircles = Table[[Black, Line[v @ circPts]], {circPts, theFPts}];

(* M(F) *)
mCircles = Table[[LightGray, Thick, Line[v @ circle]], {circle, mFPts}];

arcs = {zCircles[[firstCirc ;; lastCirc]], mCircles[[firstCirc ;; lastCirc]]};

(* POLYGONS *)

circle = .2;

nTriangles = 8; (* number of triangles *)
cosmeticRotation = Pi / 1.5; (* applied to e shapes *)

vertices = scale {0, .8 I, Sqrt[2]}; (* complex vertex points of one triangle *)

(* complex points of n triangles at increasing distance, ρ = n|e|, where w = ne *)
triPtsSet = Table[vertices + n e + q, {n, Range[1, nTriangles]}];

(* graphics objects for triangles *)
triangles = Table[{FaceForm@LightBlue, EdgeForm@Black, Polygon[v @ vertices]},
  {vertices, triPtsSet}];

(* rotate triangles out of way of arrows *)
triangles = Rotate[triangles, cosmeticRotation, v @ q];

(* points of triangles under M at w+q with increasing ρ = n|e| in F *)
mTriPtsSet = Table[Table[m[n e + q] + amplitwist[n e] vertex, {vertex, vertices}],
  {n, Range[1, nTriangles]}];
mTriangles = Table[ {FaceForm @ LightGreen, EdgeForm @ Black, Polygon[v @ mTriPts]},
{mTriPts, mTriPtsSet}];

mTriangles = Rotate[mTriangles, cosmeticRotation, v[a/c]];

polygons = {triangles, mTriangles};

(* TEXT *)

adj = .1 + .1 I;
default = 12;
small = 8;
preTwist = (a d - b c) / c^2;
preAmpli = Abs[preTwist];

txtData = {
"0", -adj, 1.5 default),
"1", 1 + adj, default),
"a", a + adj, default),
"b", b + adj, default),
"c", c + adj, default),
"d", d + adj, default),
"q", q + adj, default),
"a/c", a / c - .15 I, default),
"Figure 1. F (black) under M(z) = (az+b)/(ac+d) (gray)" <>
"\namplification = " <>ToString[Abs[preAmpli / Abs[c] ^2]] <>
"/|M'(z)|" <> "\ntwist = " <>ToString[Re[preTwist]] <> "+" <>
ToString[Im[preTwist]] <> "i/w^2 = arg(M'(z))", Re[a / c] - 1 - 2.5 pr I, default}];

(* labels for numbers on preimages *)

labels = Table[Text[Style[item[[1]], item[[3]]], v @ item[[2]]], {item, txtData}];

numbers = Table[{ToString[n], (nTriangles - (n - 1)) e + q}, {n, Range[1, nTriangles]}];

numberLabels = Table[Text[Style[item[[1]], small, Bold], v @ item[[2]]], {item, numbers}];

numberLabels = Table[Rotate[text, cosmeticRotation - Pi / 20, v @ q], {text, numberLabels}];

numberLabels = Table[Rotate[text, - cosmeticRotation], {text, numberLabels}];

(* labels for numbers on images M[shape] *)

mNumbers = Table[ {ToString[n], m[(nTriangles - (n - 1)) e + q]}, {n, Range[1, nTriangles]}];
mNLabels = Table[Text[Style[item[[1]], small, Bold], v @ item[[2]]], {item, mNumbers}];
mNLabels = Table[Rotate[text, 1.45 (Pi/2), v[a/c]], {text, mNLabels}];
mNLabels = Table[Rotate[text, -1.45 (Pi/2)], {text, mNLabels}];
labels = labels ~ Join ~ numberLabels ~ Join ~ mNLabels;

(* GRAPHICS *)
g1 = Graphics[{lines, arcs, polygons, labels, points}, Background -> White, PlotRange -> {[-2 pr, 1.1 pr], [-3.1 pr, 1.5 pr]}, ImageSize -> Large]
Figure 1. \( \mathcal{F} \) (black) under \( M(z) = (az+b)/(ac+d) \) (gray)

amplification = \( \frac{1.13281}{\rho^2} = |M'(z)| \)

twist = \( \frac{-0.378141 + 1.06784i}{w^2} = \arg(M'(z)) \)

(i) By considering orthogonal connecting vectors from one member of \( \mathcal{F} \) to an infinitesimally larger member of \( \mathcal{F} \), deduce that the amplification of \( M \) is constant on each circle of \( \mathcal{F} \). Deduce that \( |M'| \) must be a function of \( \rho \) alone.

The circles of \( \mathcal{F} \) are concentric, so orthogonal \( \epsilon \) arrows connecting \( \mathcal{F}_n \) to \( \mathcal{F}_{n-1} \) have the same length on the entire circumference of \( \mathcal{F}_n \). \( M \) preserves angles, so an orthogonal \( \epsilon \) arrow on \( \mathcal{F} \) goes to an orthogonal \( \epsilon \) arrow on \( \mathcal{F}' \).
Let \( w = M(z) = \tilde{\rho} e^{i\theta} \). We can expect \( \tilde{\rho} \) to be constant across all \( \theta \) because the circles of \( \mathcal{F} \) are concentric. We also expect \( \tilde{e} \) to be constant across all \( \theta \) for the same reason. If \( \tilde{\rho} \) were different from \( \tilde{\rho}_{\theta+\phi} \) on a circle of \( \mathcal{F} \), then orthogonal angles would not be preserved. Hence, the amplification of \( M \) is constant on each circle of \( \mathcal{F} \) and \( |M'| \) must be a function of \( \rho \) alone, as opposed to \( \rho \) and \( \theta \).

We attempt to deduce this by taking the derivative of \( M(z) \).

\[
M'(z) = \left( \frac{az+b}{cz+d} \right)' = \left( (az + b)(cz + d)^{-1} \right)'
\]

\[
= a(cz + d)^{-1} + (az + b)(-c)(cz + d)^{-2}
\]

\[
= \frac{a(cz+d)-c(az+b)}{(cz+d)^2} = \frac{acz-ad+bc}{c^2(z-q)^2} = \frac{ad-bc}{c^2(z-q)^2} \quad (1)
\]

We use (1) in (vi) and in programming twist in Figure 1.

\[
|M'(z)| = \frac{|ad-bc|}{|c|^2|z-q|^2} = \frac{|ad-bc|}{|c|^2|\rho|^2}
\]

Points a, b, c and d are constants. Hence, for \( M \) such that \( (ad-bc) \neq 0 \) and \( c \neq 0 \), \( |M'| \) depends on \( \rho \) alone.

(ii) By considering the image of an infinitesimal shape that starts far from \( q \) and then travels to a point very close to \( q \), deduce that at some point in the journey, the image and preimage are congruent.

[November 17, 2015. I think there is a problem with the question. Given the distinction between image and preimage, if the image is \( M(\mathcal{F}) \), then it does not naturally travel to a point very close to \( q \) by decreasing \( \rho \), which is a quality of the preimage, but the image may approach \( a/c \) with increasing \( \rho \). Therefore, I think the question should read “By considering the image under \( M \) of an infinitesimal shape that starts far from \( q \) and then travels to a point very close to \( q \), deduce that at some point in the journey, the image and preimage are congruent.”]

Stipulate that the image is \( M(P) \), where \( P \) is the preimage, \( z \) is a point in the preimage, and \( \rho = |z-q| = |z+d/c| \). Let \( e \) be the vector of any side of the preimage or a tangent emanating from a point on the preimage and \( \tilde{e} \) be the corresponding side or tangent on the image. \( M \) is conformal, because it is a Möbius transformation. Angles are preserved. It has amplitwist because it is conformal over a region. Since it has amplitwist, the image at any size is similar to the preimage. We have shown in (i) that the amplification \( |M'| \) depends only on \( \rho \). Furthermore, Vasco has called attention to the fact that as \( \rho \) decreases, \( \tilde{e} \) vectors increase (See (i)). In this case, we are interested in decreasing \( \rho \). When \( \rho \) is very large, \( |\tilde{e}| \) is very near zero. When \( \rho \) is very small, \( |\tilde{e}| \) is large. During the journey, which is measured by decreasing \( \rho \), \( |\tilde{e}| \) increases until \( |\tilde{e}| = |e| \), at which point \( |M'(z)| = 1 \) for all \( z \), and the images are congruent.

Discussion of Figure 1: Figure 1 illustrates these points. The triangles represent \( e \) shapes. The outermost preimage triangle (blue) corresponds to the innermost image triangle (green). Because \( M \) is conformal, these images are similar. As \( \rho \) decreases for the preimage triangles, \( \tilde{\rho} = |\tilde{z}-(a/c)| \) increases for the image triangles and they increase in size. The triangle that would be congruent with the preim-
age triangles would appear somewhere between the fifth and sixth \( \mathcal{F} \) image circle from the center. This is where \( |M'| \approx 1 \). We have calculated the twist as \( \arg(M'(z)) = \arg\left( \frac{(ad - bc)}{c^2} \frac{w^2}{2} \right) \), where \( w = z - q \), without normalization. Both the preimage and image triangles were rotated by \( \pi/2 \) around \( q \) and \( a/c \) respectively after the calculations just to get them out of the way of the \( \varepsilon \) arrows, but by the question, they can be situated at any \( \arg(z) \) and \( \arg(M(z)) \).

Vasco also pointed out that the twist can be rewritten as \( \arg(A/w^2) \), where \( A = (ad - bc)/c^2 = ge^{i\alpha} \), so \( \arg(A/w^2) = -2\phi + \alpha \) and \( \phi = \arg(w) \). We can also see by Figure[1], p. 124, and by obtaining step (iii) in (3) on the same page, that \( M(z) \) has the rotation \( \arg(-A/w) = -(\alpha - 2\phi) = -2\phi + \alpha + \pi \). (Step 4 would not alter the rotation or twist.) This shows that the twist of \( M \) reverses the direction of the arrows.

(iii) Combine the above results to deduce that there is a special member \( I_M \) of \( \mathcal{F}^{-} \) such that infinitesimal shapes on \( I_M \) are mapped to congruent image shapes on the image circle \( M(I_M) \). Recall that \( I_M \) is called the isometric circle of \( M \).

\( I_M \) is the member at which \( |M'(z)| = 1 \) for all \( z \) in \( I_M \).

(iv) Use the previous part to explain why \( M(I_M) \) has the same radius as \( I_M \).

The amplification depends only on \( \rho \). Since \( |\tilde{\epsilon}| = |\epsilon| \), it must be the case that \( M'(z) = M'(w) = 1 \), which implies that \( (w = M(z)) = z \), and since \( \rho = |z-q| \) and \( \tilde{\rho} = |w-q| \), it follows that \( \tilde{\rho} = \rho \).

(v) Explain why \( I_{M^{-1}} = M(I_M) \).

\[ I_M = M(I_M), \text{ from (iv), because they have the same radius.} \]

\[ I_{M^{-1}} = M^{-1}(I_M) = M^{-1} \circ M(I_M) = I_M = M(I_M) \]

(vi) Suppose that \( M \) is normalized. Using the idea in Ex. 16, show that the amplification of \( M \) is

\[ |M'(z)| = \frac{1}{|c|^2 \rho^2} \]

From Ex. 16, the amplification \( a = |f'(z)| \).

\[ |M'(z)| = \frac{|ad-bc|}{|c|^2 \rho^2} \quad \text{from (i)} \]

If \( M \) is normalized, \( ad-bc = 1 \). Then, letting \( k = \pm \sqrt{ad - bc} \), there is a \( |\frac{c}{k}|^2 \) in the denominator, so the \( k \) goes to 1.

\[ |M'(z)| = \frac{1}{|c|^2 \rho^2} \]