Needham, Chapter 3, Exercise 3, p. 181
Gary Palmer, March 12, 15, 16, 2015

3. Let S be a sphere, and let p be a point not on S. Explain why \( I_S(p) \) may be constructed as the second intersection point of any three spheres that pass through p and are orthogonal to S. Explain the preservation of three dimensional symmetry in terms of this construction.

In two dimensions, from page 129, “The inverse \( \tilde{z} \) of z in K is the second intersection point of any two circles that pass through z and are orthogonal to K.” From p. 135 we have that any two orthogonal spheres intersect a plane \( \Pi \) on which two orthogonal great circles pass through their centers. “Then \( S_1 \) and \( S_2 \) are orthogonal if and only if \( C_1 \) and \( C_2 \) are orthogonal.” Any sphere that passes through p not on sphere S and is orthogonal to S has a great circle that is orthogonal to a great circle in S and has a second intersection point with S at \( \tilde{p} \). Since all three spheres \( S_1, S_2, \) and \( S_3 \), of the problem pass through p and are orthogonal to S, they all share a second intersection point \( \tilde{p} \). Hence, \( I_S(p) \) may be constructed as the second intersection point of any three spheres that pass through p and are orthogonal to S.

Explain the preservation of three dimensional symmetry in terms of this construction.

From (13), we have

Let \( K \) be a plane or sphere, and let \( a \) and \( b \) be symmetric points with respect to \( K \). Under a three-dimensional reflection in any plane or sphere, the images of \( a \) and \( b \) are again symmetric with respect to the image of \( K \).

By the paragraph preceding this quote, this use of the word “reflection” refers to inversion in a sphere. To explain this in terms of our construction from orthogonal spheres and center planes, orthogonal spheres \( S_i \) may be drawn on S with center points \( q_i \) at any angle on the xy axis, proceeding, say, from x at 0 to 2Pi and any angle on the yz axis from -Pi to Pi. Every such sphere shares a center plane \( \Pi \) with sphere S and center \( q \), and it intersects the two points \( \rho_i \) and and its inversion in \( \tilde{\rho}_i \) on the great circle \( K_i \) in S. Every \( \rho_i \) and \( \tilde{\rho}_i \) on \( \Pi \) is symmetric with respect to \( K_i \) corresponding to each \( S_i \). Hence, three dimensional symmetry is preserved with respect to S.

Regarding the construction in which three orthogonal spheres share a single \( p \) and \( \tilde{p} \) pair, they all preserve symmetry at \( p \) and \( \tilde{p} \) under reflection on S and each one preserves symmetry under reflection on the plane defined by the center points \( c_1, c_2, \) and \( c_3 \), which can also be written with the dot product.

\[
[c_i - (\tilde{\rho}_i - p)/2].(\tilde{\rho}_i - p) = 0
\]

The manipulate panel displays a base sphere and three orthogonal spheres in three colors. The base sphere is \( S_K \), shown in red. Notice that all the orthogonal spheres pass through the points \( p \) and \( \tilde{p} \). Since the angle of the tangent T at \( p \) is randomly chosen (within limits chosen to enhance the viewing angles) in three dimensions for each orthogonal sphere, the orthogonal spheres are rotated by random amounts around the axis defined by \( p\tilde{p} \) (colinear with \( \tilde{\rho}q \) and \( pq \)) as shown by the angles of their great circles, except for the red sphere labeled \( S_1 \), which is unrotated. It’s great circle lies in the plane of inversion with the great circle \( K \) in \( S_K \), which is on the xy plane. All the arrows also lie on this plane and apply only to the red orthogonal sphere. The three red points at the centers of the three orthogonal
spheres lie in a plane intersecting the segment \( pp \) at its midpoint and orthogonal to it. When the red orthogonal sphere appears to be large, it is because the tangent \( T \) at \( p \) happens to make a very small angle with the axis \( pp \). Sometimes, the orthogonal spheres are so large that they extend beyond the viewing space, but it is still possible to determine that they pass through \( p \) and \( pp \) by following the paths of their great circles. In order to see features and angles, rotate the entire image by dragging with the mouse, or just click for a new set of spheres. For example, the plane of inversion on the xy plane can be seen only by rotating the image with the mouse. Drag across the middle orthogonally to the \( pp \) axis.

From p. 131, para. -1, we have

"**Inversion in a circle is an anticonformal mapping.**"

"To see this, first look at [9]. This illustrates the fact that given any point \( z \) not on \( K \), there is precisely one circle orthogonal to \( K \) that passes through \( z \) in any given direction. [Given the point and the direction, can you think of how to construct this circle?]"

All circles that pass through a point \( p \) and its inversion \( p \) are orthogonal to \( K \). We find the center of \( K \) by drawing orthogonals from the directions at both \( p \) and \( p \). We draw the second orthogonal as a ray from \( p \) and see that it intersects the center found by the first method.

Our plan is to proceed as we did for the square circles in the plane. Then we will plot the spheres and rotate the circles and spheres.

1. Plot a circle \( K_1 \) on the Argand plane.
2. Plot point \( p \) anywhere on the plane not on \( K \).
3. Calculate the inversion \( p \).
4. Pick a direction on \( p \) using a random angle \( \theta = 3\pi/8..\pi \), and construct or plot a line \( T \) in this direction from \( p \). Let this be the tangent of the orthogonal circle.
5. Construct or plot a line segment \( T' \) orthogonal to \( T \) at \( p \).
6. Construct or plot the line segment \( \tilde{T} \) from \( p \) as the line of length \( |T| \) having the same acute angle with \( p \) as does \( T \).
7. Construct or plot the line segment \( \tilde{T}' \) orthogonal to \( \tilde{T} \) at \( p \).
8. Find the intersection point \( c \) of \( T' \) and \( \tilde{T}' \). This requires solving the system of two equations describing the lines \( T' \) and \( \tilde{T}' \) for \( c \).
9. Plot the orthogonal circle \( K_{\perp} \) with radius \( \text{Abs}(c-p) \) at center \( c \).
10. Plot the orthogonal sphere on \( c \) with radius \( \text{Abs}(c-p) \).
11. Repeat from 4 with a second sphere and rotate the second sphere and circle by a random angle \( \theta = \pi/4..3\pi/4 \).
12. Repeat from 4 with a third sphere and rotate the third sphere and circle.

The Argand plane is nested within three dimensions by substituting the real and imaginary parts of complex numbers for \( x \) and \( y \). The plot has viewpoint \( \{0, 0, 2.8\} \).

\[ \text{In}[207]:= \text{Clear}["Global`*" ]; \]
(* GLOBALS *)

pr = 2.5; (* plot range basis *)
q = 0;

r = 1; (* radius of circle K and sphere S *)
gSRecs = {}; (* records for orthogonal spheres *)

ahSz = .02; (* size of arrow heads *)
ptSz = .012; (* point size *)
ptColor = Red;
kSphOpac = .3;
(* opacity of S_k and unrotated orthogonal sphere based on xy plane *)
sphOpac = .15; (* opacity of orthogonal spheres *)
arwColor = Black;
circleColor = EdgeForm[Gray];
sColor = Red; (* color of base Sphere, S_k *)

ipT = 1;
idT = 2; itD = 3; izTInv = 4; izDTInv = 5; ic = 6;irs = 7;

(* 3. Pick a direction on p;
let this be the tangent of the orthogonal circle;
plot the tangent,T. *)

(* FUNCTIONS *)

(* imaginary to vector *)

v[p_, z_] := {Re[p], Im[p], z};

(* The inverse function *)

ik[z_, q_, r_] := (r^2/((Conjugate[z] - Conjugate[q])) - q;

(* angle between T and p *)

θ[p_, pT_] := Arg @ pT - Arg @ p;

(* angle of T *)

argTInv[p_, θ_] := Arg @ p + Pi - θ;

(* CALCULATED POINTS *)

(* MODULE solveForC[] *)
solveForC[i_, p_, dirT_] :=

Module[
  {pT, dT, m, sb, b, eqDT, zTInv, zDTInv, mDTInv, bDTInv, sb2, eqDTInv, c, rs},

(* End of code *)
pT = Exp[I dirT]; (* unit length *)

dT = pTI;

(* SOLVE FOR \( \tilde{T} \), \( \tilde{T}' \), and c. *)

(* Find equation for \( T' \) *)

m = Im[dT] / Re[dT];
sb = Solve[Im[p] == m Re[p] + b, b];
eqDT = ((y == m x + b) /. sb);
Print["T' \rightarrow ", eqDT];

(* Plot \( \tilde{T} \) and \( \tilde{T}' \) *)

(* complex vector of \( \tilde{T} \) *)

zTInv = Exp[I argTInv[p, \( \Theta \)[p, pT]]];

zDTInv = I zTInv; (* \( \tilde{T}' \) *)

(* Find equation for \( \tilde{T}' \) *)

mDTInv = Im @ zDTInv / Re @ zDTInv;
sb2 = Solve[Im[invP] == mDTInv Re[invP] + bDTInv, bDTInv];
eqDTInv = ((y == mDTInv x + bDTInv) /. sb2);
Print["\( \tilde{T}' \) \rightarrow ", eqDTInv];

s = Solve[eqDT[[1]] && eqDTInv[[1]], \{x, y\}] // Flatten;
c = v[(x + y I), 0] /. s;
Print["c: ", c];
rs = Norm[c - v[p, 0]]; 
Print["r[[", i, "]]: ", rs];

{pT, dirT, dT, zTInv, zDTInv, c, rs}
]

(* end module *)

(* Make all the sphere records. *)
sphereRec = {pT, dT, dirT, zTInv, zDTInv, c, rs} *)

ipT = 1; iDirT = 2; idT = 3; izTInv = 4; izDTInv = 5; ic = 6; irs = 7;

mkOrthoSphereRecs[nOSpheres_] :=
Module[{recs, dirT},
  recs = {};
  For[i = 1, i <= nOSpheres, i++,
    dirT = RandomReal[{3 Pi/8, Pi}];
    recs = Append[recs, solveForC[i, p, dirT]]
  ];
  recs
];

(* GLOBAL GRAPHICS OBJECTS *)

(* K *)
circleK = {Opacity[.05], circleColor, Cylinder[{{v[q, -.01], v[q, .01]}, r}}];
sphereS = {Opacity[.3], sColor, Sphere[v[q, 0], r]};

(* p *)
ptP = {PointSize[ptSz], Point[v[p, 0]]};
invPpt = {Black, PointSize[ptSz], Point[v[invP, 0]]};
cK = {PointSize[ptSz], Point[v[q, 0]]}; (* center of K *)
lineCInvP = {Dashed, Gray, Line[{{v[q, 0], v[invP, 0]}]}];
sGrObjects = {circleK, sphereS, ptP, invPpt, cK, lineCInvP};

(* Plot the inverse p. *)

(* GRAPHICS OBJECT FUNCTIONS *)

lnT[sphRecs_, i_] :=
  {arrowColor, Arrowheads[ahSz], Arrow[{v[p, 0], v[p + sphRecs[[i, ipT]], 0]}]};

lnDT[sphRecs_, i_] :=
  {arrowColor, Arrowheads[ahSz], Arrow[{v[p, 0], v[p - sphRecs[[i, idT]], 0]}]};

(* ipT = 1;
iDirT = 2; idT = 3; izTInv = 4; izDTInv = 5; ic = 6; irs = 7; *)

theTInvLn[sphRecs_, i_] := {arrowColor, Arrowheads[ahSz],
  Arrow[{v[invP, 0], v[invP + sphRecs[[i, izTInv]], 0]]};

dTInvLn[sphRecs_, i_] := {arrowColor, Arrowheads[ahSz],
  Arrow[{v[invP, 0], v[sphRecs[[i, izDTInv]] + invP, 0]]};

cPt[sphRecs_, i_, θ_] := {Opacity[1], ptColor, PointSize @ ptSz,
  Rotate[Point[sphRecs[[i, ic]]], θ, v[invP, 0], v[q, 0]]};

orthoCircle[sphRecs_, i_, θ_, _] := Module[{c, rs, cyl},
  c = sphRecs[[i, ic]]; rs = sphRecs[[i, irs]]; cyl = Cylinder[
    {{c[[1]], c[[2]], -.01}, {c[[1]], c[[2]], .01}}, rs ];
  {Opacity[.05], circleColor, Rotate[cyl, θ, v[invP, 0], v[q, 0]]
  }
];

orthoSphere[sphRecs_, i_, θ_, color_, opac_] :=
  {Opacity[opac], color, Rotate[ Sphere[ sphRecs[[i, ic]], sphRecs[[i, irs]]], θ, v[invP, 0], v[q, 0]]};

orthoSphereObjs[sphRecs_, i_, θ_, color_, opac_] := {
  cPt[sphRecs, i, θ],
  orthoCircle[sphRecs, i, θ],
  orthoSphere[sphRecs, i, θ, color, opac]
};

indicatorArrows[sphRecs_, i_] := {
  theTInvLn[sphRecs, i],
dTInvLn[sphRecs, i],

lnT[sphRecs, i],

lnDT[sphRecs, i]

);

(* TEXT, LABELS, ETC. *)

labels[sphRecs_, i_] := {
   "p", v[p - .1 -.1 I, 0]},
   "\(p\)"
   v[invP + .05 + .2 I, 0]},
   "q", v[q - .1 -.1 I, 0]},
   "T", v[p + sphRecs[[i, ipT]] + .1 I, 0]},
   "T'", v[p - sphRecs[[i, idT]], 0]},
   "\(T\)", v[invP + sphRecs[[i, izTInv]] + .05 I, 0]},
   "\(T'\)", v[invP + sphRecs[[i, izDTInv]] + .05 I, 0]},
   "c", v[sphRecs[[i, ic, 1]] + (sphRecs[[i, ic, 2]] - .2) I, 0]},
   "K", v[r / 2 - I, 0]},
   "S_k", v[g - r / 2 - (r / 2) I, .5]},
   "S_\perp", v[sphRecs[[i, ic, 1]] +
    sphRecs[[i, ic, 2]] I + sphRecs[[i, irs]] / 2, sphRecs[[i, irs]] / 2}]

caption = Style[
   "\nFigure 1. Three orthogonal spheres S_\perp on S\nbased on arbitrary p and T_1"",
   Bold, Black, 14];

labelTxt[sphRecs_, i_] :=
   Text[Style[H[[1]], Bold, 10], H[[2]] ] & /@ labels[sphRecs, i];

(* GRAPHICS *)

Manipulate[

   n = False;

   (* FIND p, pT, T, T' *)
   randomSign := If[RandomInteger[] = 1, 1, -1];
   p = randomSign RandomReal[{r / 3, r / 2}] + RandomReal[{r / 3, r / 2}] I;
   (* \(p\) *)
   invP = IK[p, q, r];
   gSRecs = mkOrthoSphereRecs[3];
   Graphics3D[{sGrObjects, labelTxt[gSRecs, 1],}
orthoSphereObjs[gSRecs, 1, 0, sColor, kSphOpac],
orthoSphereObjs[gSRecs, 2, RandomReal[{Pi/4, 3 Pi/4}], Green, sphOpac],
orthoSphereObjs[gSRecs, 3, RandomReal[{Pi/4, 3 Pi/4}], Yellow, sphOpac],
indicatorArrows[gSRecs, 1],
Background -> White,
PlotRange -> {{-pr, pr}, {-pr, pr}, {-pr, pr}},
ViewPoint -> {0, 0, 2.9}, Boxed -> False, ImageSize -> Medium],

Button["Get new spheres", n = True],
FrameLabel ->
  Style["Figure 1. Three orthogonal spheres $S_i$ on $
\vec{S}$\nbased on arbitrary $p$ and $T_i$",
    Bold, 16],

TrackedSymbols -> {n}, SaveDefinitions :> True

]