Unit 2 – Problem Solving, Reasoning and Mathematical Modeling  
Section C – Linear and Exponential Equations  
Topics: Linear, exponential and log functions.

Linear Equation:
- All linear equations have the form \( y = mx + b \)
- The letter \( m \) is the slope of the line, \( \frac{\text{rise}}{\text{run}} = \frac{\text{change in vertical}}{\text{change in horizontal}} \). It can be positive, negative or zero. It can also be very large or very small.
- The letters \( x \) and \( y \) are variables, meaning they vary or change along the line. At least one of them must be nonzero. Together they represent the ordered pairs \((x, y)\).
- The letter \( b \) represents the \( y \) (or horizontal) intercept, this is where the line crosses the horizontal axis.

The Disguises of a Linear Equation:
- The linear equation need not be in the form \( y = mx + b \) to be linear.
- Recall that to be a line it has to have an \( x \) or a \( y \) (not necessarily both) and they both need to only have a power of 1. If there is a squared term it is not a line (in other words, \( y = x^2 + 2 \) is not a line).
- Which of these are equations of lines?
  - \( 2x + y = 7 \)
  - \( 2y = 8 \)
  - \( 3x – 6 = y \)
- If there is no \( x \) in the equation, the line is of the form \( y = b \). It is horizontal.
- If there is no \( y \) in the equation, the line is of the form \( x = b \). It is vertical (not a function).
- Every other line will have either a positive or negative slope \( (m) \).

The Slope of a Line:
- \( m \), the slope of the line can be positive or negative.
- It is a measure of \( \frac{\text{rise}}{\text{run}} = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} \).
- What is so special about the slope of vertical and horizontal lines?
- Let's find the slope of the line between two points \((x_1, y_1)\) and \((x_2, y_2)\).
- \( m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_2 - x_1} \)
- Find the slope of the line through \((a, 3a+1)\) and \((a+h, 3(a+h)+1)\).
  - \((a, 3a+1)\) and \((a+h, 3a+3h+1)\)
  - \( m = \frac{\Delta y}{\Delta x} = \frac{3a+1-(3a+3h+1)}{a-(a+h)} = \frac{-3h}{-h} = 3 \)
**Finding the Equation of a Line:**
- If you have two points or one point and the slope you can find the equation of a line.
- **Find the equation of the line through the points (1,2) and (4,8)**
  
  \[
  m = \frac{8 - 2}{4 - 1} = \frac{6}{3} = 2
  \]
  
  \[y - 2 = 2(x - 1) \Rightarrow y = 2x\]

**Determining Linear Relationships:**
- There are several ways to determine a relationship is linear
  - The graph is a line
  - The equation is given and it is linear in form
  - You are told something increases (or decreases) by a constant or fixed amount
  - The change in \( y \) is proportional to the change in \( x \) (i.e. \( \Delta y = m \cdot \Delta x \)).

**Rules for Exponents:**

\[
\begin{align*}
  b^x b^y &= b^{x+y} \\
  (b^x)^y &= b^{xy} \\
  b^{-x} &= \frac{1}{b^x}
\end{align*}
\]

**Form of the Exponential Function:**
- \( y = kb^t \)
- Variables are \( y \) and \( t \), \( y \) depends on \( t \).
- \( b \) is the base (\( b > 0, \ b \neq 1 \)).
- \( k \) is the initial quantity (when \( t = 0 \)).
- A commonly used base is \( e = 2.7182818284… \)

**Graphs of Exponentials:**
- \( y = kb^t \)
  - If the base is greater than 1, you have exponential growth.
  - If the base is less than 1, you have exponential decay.
  - If \( k > 0 \), the function will always be positive. It will tend to 0 (for decay) and infinity (for growth).
  - If \( k < 0 \), the function will always be negative (it is flipped upside down).
  - The y intercept is \( k \).

**Half Life andDoubling Time:**
- So \( b \) is the base value in the equation \( y(t) = kb^t \), and the rate of growth (or decay) is sometimes given as \( r \).
- Note that \( r \) is usually given as a percentage, and you must convert it to decimal before using
  \( b = 1 + r, \ r = b - 1 \)
  
  \( r \) can be negative or positive, but \( b \) must be positive and not equal to 1
• We can also talk about the exponential form of the equation as $y(t) = P_0 e^{kt}$. Here $k$ is called the exponential growth rate. Be careful of the difference between this $k$ and the plain growth rate ($r$) given above.

• **Half life** is the amount of time it takes for a substance to decay to half of its original quantity. It represents exponential decay. The half life and decay rate are obviously related:

$$\frac{1}{2}k = k \cdot b^T \implies T = \frac{\ln(0.5)}{\ln b} \quad \text{or} \quad b = \left(\frac{1}{2}\right)^{1/T}$$

• **Doubling time** is the amount of time it takes for a substance to double its original quantity. It represents exponential growth. The doubling time and growth rate are obviously related:

$$2k = k \cdot b^T \implies T = \frac{\ln(2)}{\ln b} \quad \text{or} \quad b = 2^{1/T}$$

**Log Functions:**

• We want to undo the exponential function $b^y = x$

• This is true iff $y = \log_b x$

• $y$ is the exponent, $b$ is the base, and $x$ is the argument

• So the log function is the inverse of the exponential function

• What are our conditions on $x$, and $b$?

**Working with Logs (Changing forms):**

• **Example.** $\log_{10} 1000 = 3 \quad 10^3 = 1000$

• **Example.** $\log_{10} \frac{1}{100} = ? \quad 10^y = \frac{1}{10^2}, \quad y = -2$

• **Example.** $\log_{0.5} 16 = y \quad 0.5^y = 16, \quad y = -4$

• **Example.** Find $\ln e^{-5} = y$

  $e^y = e^{-5}$

  $y = -5$

• **Example.** Convert $e^{-t} = 4000$ to log

  $\log_e 4000 = -t$

**Special Log Bases:**

• Log base $e$ is natural log (written $\ln$)

• Log base 10 is common log (written $\log$)

• These will be the only two on your calculator. So if you need to calculate say, $\log_a 2$, you have to use the change of base formula $\log_b M = \frac{\log_a M}{\log_a b}$.

  $\log_4 2 = \frac{\ln 2}{\ln 4} = \frac{\log 2}{\log 4} = 0.5$

• **Example, page 369 number 74.** Find $\log_{5,3} 1700$

  $\log_{5,3} 1700 = \frac{\ln 1700}{\ln 5.3} \approx 4.46$