History of Math: - Spring 2004

Special Topic: Math as it relates to art and music

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Lesson Plan

Math relating to art:

In this section, I will touch upon the following:

- The Golden Section and art.
- Perspective and projective geometry.
- Kaleidoscopes.
- Fractals in art.

Math relating to music:

In this section, I will touch upon the following:

- The Golden Section and music.
- Mathematics of musical instruments.
- Fractals in music.
The Golden Section and art:

The Golden Section is also called as the golden number or golden ratio, golden mean or the divine proportion. Two numbers are both called by these various terms. The two numbers are 1.6180339887... and 0.6180339887. A mathematical definition of the Golden Ratio is a number whose square is obtained by adding 1 to itself.

\[ \phi^2 = \phi + 1. \]

These two numbers are denoted as Phi and phi [1-2].

Artists have often used the Golden Section as it leads to the picture design being more pleasing to the beholder’s eye.

On investigation, it has been found that many paintings and sculptures adhere to the Golden Ratio in their design. Many of Leonardo da Vinci’s paintings have incorporated in them the Golden Ratio in the forms of Golden Rectangles (rectangles whose length to width ratio is nearly equal to Phi.). Another artist whose works include a lot of Golden Rectangles in them is Piet Mondrian. Here are a few examples of their works [10]:

Mona Lisa
Many books on oil painting and watercolor point out that it is better to position objects not in the center of the picture but to one side (about one-third) of the way across, and to use lines which divide the picture into thirds. This seems to make the picture design more pleasing to the eye and relies again on the idea of the golden section being "ideal" [1-2].

Other artists who have utilized Golden Sections in their paintings and sculptures are as follows: Michelangelo, Georges Seurat, Paul Signac and Albrecht Durer.

This section would be enhanced if one were to have the class discover the different Golden Rectangles and Golden Sections in different art works. However, I decided not to include this exercise because it would be more suitable if the class were an hour long and not half an hour, as it is presently.
Perspective and Projective Geometry

Perspective was introduced as part of the Renaissance determination to return to the Greek origins of science. It was a complete science of vision, encompassing not only the nature and behavior of light, but the anatomy and functioning of the human eye as well. The theory of perspective describes how to project a three-dimensional object onto a two-dimensional surface.

Before the advent of perspective, it was generally accepted that the function of art was not naturalistic representation, but rather the expression of spiritual power. However, by the 13th century, naturalistic style started to appear in paintings. The first person credited with using this style was Giotto di Bondone (1266 – 1337). He however did not master the art of linear perspective correctly as one can see from the painting below [3]

Joachim comes to the shepherds’ hut”

As one can see in the painting above, Giotto has drawn the human and animal figures to scale but has shown the landscape very small. However, in his work at the near end of his career, he appears to have grasped the concept of linear perspective better.
The person credited with inventing correct mathematical rules for perspective was Filippo Brunelleschi (1377-1446). He wasn’t a painter but a goldsmith. However, no works of his on perspective have survived. He is remembered for designing buildings and at-least two paintings on correct perspective. The person to have first written a written treatise on perspective was Leon Battista Alberti (1404 – 1472). He actually wrote two books on perspective, the first was meant for scholars while the second one was aimed at a general audience (and was dedicated to Brunelleschi). Alberti defined a painting as follows:

“A painting is the intersection of a visual pyramid at a given distance, with a fixed centre and a defined position of light, represented by art with lines and colours on a given surface.” [3].

Alberti gave the background on the principles of geometry, and on the science of optics. He also set up a system of triangles between the eye and the object viewed which define the visual pyramid referred to above. He gave a precise concept of proportionality, which determined the apparent size of an object in the picture relative to its actual size and distance from the observer.

The most mathematical work on perspective was written by Piero della Francesca (1412 – 1492). Besides being a talented painter, he was also a highly competent mathematician. Piero wrote three treatises: “Abacus treatise” (Trattato d’abaco), “Short book on the five regular solids” (Libellus de quinque corporibus regularibus), and “On perspective for painting” (De prospectiva pingendi).

Piero’s perspective treatise was concerned not with ordinary natural optics, but with what was known as “common perspective” (perspectiva communis), the special kind used by painters. Piero felt that this new part of perspective should be seen as a legitimate extension of the older established science.

Leonardo da Vinci was another artist who studied the art of perspective and also studied the optical principles of the eye to create reality as seen by the eye. Leonardo distinguished two different types of perspective: artificial perspective which was the way that the painter projects onto a plane which itself may be seen foreshortened by an observer viewing at an angle; and natural perspective which reproduces faithfully the relative size of objects depending on their distance. In natural perspective, Leonardo
correctly claims, objects will be the same size if they lie on a circle centered on the observer. Then Leonardo looked at compound perspective where the natural perspective is combined with a perspective produced by viewing at an angle. By about 1500, Dürer introduced perspective in Germany. He published *Unterweisung der Messung mit dem Zirkel und Richtscheit* in 1525, the fourth book of which contains his theory of shadows and perspective. Geometrically his theory is similar to that of Piero but he made an important addition stressing the importance of light and shade in depicting correct perspective. Another contribution to perspective made by Dürer in his 1525 treatise was the description of a variety of mechanical aids, which could be used to draw images in correct perspective [3-4].

Other contributors to the area of perspective were Federico Commandino, Wentzel Jamnitzer, Daniele Barbaro, Egnatio Danti and Giovanni Battista Benedetti.

The earliest surviving painting, which used linear perspective, is *Masaccio's Trinity* [11]
Projective geometry was another area, which developed due to perspective. The main person contributing to this was Girard Desargues (1591 – 1661). Desargues wrote his most important work, the treatise on projective geometry, when he was 48. It was entitled “Rough draft for an essay on the results of taking plane sections of a cone” (*Brouillon proiect d’une atteinte aux evenemens des rencontres du cone avec un plan*). In the course of his work, he became recognized as the first mathematician to get the idea of infinity properly under control, in a precise, mathematical way. Others who contributed to projective geometry were Humphry Ditton and Phillippe de La Hire (1640-1718) [3-4].

Many artists also deliberately misused perspective and ended up drawing what are aptly termed as “Impossible Figures”. The chief artists who did this were William Hogarth (1697-1764) and Maurits Escher (1898-1972). Here is an example of such a work [4]:

*Escher’s Waterfall.*
Kaleidoscopes

The kaleidoscope is a beautiful application of geometry. It was invented by Sir David Brewster, a Scottish scientist, in 1816. He named his invention after the Greek words, kalos or beautiful, eidos or form, and scopus or watcher. So kaleidoscope means the beautiful form watcher. Brewster's kaleidoscope was a tube containing loose pieces of colored glass and other pretty objects, reflected by mirrors or glass lenses set at angles, that created patterns when viewed through the end of the tube [5-6].

Kaleidoscopes are basically built from mirrors. Here are a few designs [6]:

**The Three Mirror Kaleidoscope**
*makes an endless field of patterns*

**The Two Mirror Kaleidoscope**
*makes a single, circular design*
Different types of kaleidoscopes are as follows [6]:

1. **The Chamber Kaleidoscope**
   - Has an enclosed object case with free-tumbling jewels, dichroic glass, beads or other objects.

2. **The Liquid Chamber Scope**
   - Has an object case filled with liquid (usually glycerine) jewels, dichroic glass, beads or other objects.

3. **The Wheel Scope**
   - Has one or more wheels at its objective end. These may contain glass, translucent agates, pressed flowers, beads, jewels or other objects.

4. **The Refillable Scope**
   - Features a removable object chamber. You can change the contents & experiment with your own assortment of colors & objects.

5. **The Teleidoscope**
   - Makes its image with prisms & lenses alone. Anything you view appears multiplied. The teleidoscope transforms the whole world into a riot of kaleidoscopic images.
Fractals in art

A fractal is a rough or fragmented geometric shape that can be subdivided in parts, each of which is (at least approximately) a reduced-size copy of the whole. Fractals are generally self-similar and independent of scale [7]

Generation of art by machines is very intriguing. Modern computers can generate and evaluate fractal patterns. These patterns are the product of mathematical feedback loops (iterated maps and iterated function systems). Computers can be used to generate fractal patterns, which are visually attractive to human beings [7.1]

Such fractal patterns are as follows [8]:

Fractal art has a huge future in the field of fashion. It can be used on casual wear, office wear and even posters, key chains, cards etc. In the future, it is expected that fractal art be used to make camouflage uniforms for soldiers. Fractals are already in use in the entertainment industry viz. movie special effects [7.2].
The Golden Section and music

Mozart and the Golden Section:

Mozart used the Golden Section extensively. An analysis of his sonata’s revealed that many of them divide into two parts exactly at the Golden Section point. This is supposed to be a conscious choice on his part [1].

The Golden Section in Beethoven's Fifth:

The famous opening "motto" appears not only in the first and last bars (bar 601 before the Coda) but also exactly at the golden mean point 0·618 of the way through the symphony (bar 372) and also at the start of the recapitulation which is phi or 0·382 of the way through the piece [1].

The Golden String as Music

The Golden String is a fractal string of 0s and 1s that grows in a Fibonacci-like way as follows:

```
1
10
101
10110
10110101
1011010110110
101101011011010110101
...
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After the first two lines, all the others are made from the two latest lines in a similar way to each Fibonacci numbers being a sum of the two before it. Each string (list of 0s and 1s) here is a copy of the one above it followed by the one above that. The resulting infinitely long string is the Golden String or Fibonacci Word or Rabbit Sequence [1, 7.3].
Mathematics of musical instruments

The instruments, which have been studied mathematically, are the Flute, the Drum, the violin and the piano. Each instrument alone can be accommodated for more than an hour. One utilizes partial differential equations and boundary conditions to do the mathematical study of these instruments [9].

I won’t attempt to list any of the methods used to do this as that would turn out to be too voluminous and should be treated as a separate topic on their own merit.

Some designers have also actually come out with mathematical methods of constructing these music instruments.
Fractals in music

Fractal patterns have appeared in music for quite some time. On investigation, it has been found that fractal structures have existed in many baroque and classical works (especially Bach) [7.3]. One specialty of music that has a particular fractal pattern in them is that the music will sound the same no matter at what speed they are reproduced. Other music sounds that have another kind of fractal pattern in them have the specialty that their pitch continuously rises while their frequency doesn’t. The rabbit sequence is yet another fractal pattern that has been employed in music [7.3].
References

