Transcendental vs. Algebraic Numbers

History and Definitions

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In mathematics, an **algebraic number** is any real or complex number that is a solution of a polynomial equation of the form

\[ a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0 \]

where \( n > 0 \) and every \( a_i \) is an integer, and \( a_n \) is nonzero.

All rational numbers are algebraic because every fraction \( a / b \) is a solution of \( bx - a = 0 \).

Some irrational numbers such as \( \sqrt{2} \) (the square root of 2) and \( \sqrt[3]{3} / 2 \) (the cube root of 3 divided by 2) are also algebraic because they are the solutions of \( x^2 - 2 = 0 \) and \( 8x^3 - 3 = 0 \), respectively.

But not all real numbers are algebraic. Examples of this are pi and e.

• Adrien-Marie Legendre conjectured that I could not be the root of any polynomial equation with rational coefficients, and as a result, the existence of numbers which 'transcend algebra' gradually came to be known as transcendental numbers (2, pg 261)

• Hermite proved e to be transcendental in 1873, and Lindemann proved to be transcendental in 1882 (1, page 1)

• How many transcendental numbers are there, and can they be counted? In 1844 Liouville proved any number of the form

\[ \frac{a_1}{10^1} + \frac{a_2}{10^{2^1}} + \frac{a_3}{10^{3^2}} + \frac{a_4}{10^{4^3}} + \ldots \]

where \( a_i \) is any natural number up to 9 (2, pg 261)

History of the Number e

• It first appeared in 1618 in an appendix probably by Oughred, it was a table giving natural logs (1, pg 1)

• In 1624, Briggs gave a numerical approximation to the base 10 log of e but didn’t mention e itself (1, pg 1)

• In 1647 Saint Vincent computed the area under a rectangular hyperbola, but probably didn’t relate this to logs, or e (1, pg 1)

• In 1661 Huygens related the rectangular hyperbola \( yx = 1 \) and the log (1, pg 1)

• Mathematicians were slowly understanding the relation that the area under the rectangular hyperbola from 1 to e is 1, which makes e the base of natural logs (1, pg 1)

• Huygens defined the curve \( y = k a^x \), which he called logarithmic, and he estimates \( \log_{10} e \) to 17 places (1, pg 1)

• In 1668 Mercator published *Logarithmotechnia* where he provides the series expansion of log(1+x), and coins the phrase natural log with a base of e (1, pg 1)

• Bernoulli first discovered e while looking at compound interest, and used the Binomial theorem to say that the limit of \( (1+1/n)^n \) was between 2 and 3. He made no connections to logs (1, pg 1)

• The reason for the disconnect between logs and exponents is that logs were not thought of as functions. They were first considered only as an aid in calculations (1, pg 1)

• Bernoulli was the first to understand the log function is the inverse of the exponential function (1, pg 1)
• Gregory also made the connection between logs and exponents in 1684, but may not have been the first (1, pg 1)
• In 1690 Leibniz used the letter $b$ to identify what we now call $e$ (1, pg 2)
• Bernoulli began to study the calculus of the exponential function in 1697 (1, pg 2)
• Euler is credited with using the letter $e$, probably because it was the next vowel (1, pg 2)
• In 1731 Euler published *Intro to Analysis* and showed that $e = \lim_{n \to \infty} \sum_{k=0}^{n} \frac{1}{k!}$, and that it is also equal to $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$. He gave an approximation to 18 places (2, pg 2)
• Euler also connected $e$ to sin and cos using DeMoivre’s formula (1, pg 2)
• Euler also gave the continued fraction expansions of $e$ and noticed a pattern, which was the first identification that $e$ was irrational (1, pg 2)
• Shanks in 1854 was the first to estimate $e$ to many places, which was verified by Glaisher to 137 places, and then corrected and extended by Shanks to 205 places (1, pg 3)
• In 1873 Hermite proved $e$ is not algebraic (1, pg 3)
• Most people agree the Euler was the first to prove $e$ is irrational (1, pg 3)

**History of the Number $\pi$**

• The fact that the ratio of the circumference to the diameter of a circle is constant has been known for a long time (2, pg 1)
• Egyptian and Mesopotamian values of pi were estimated by $25/8 = 3.125$ (2, pg 1)
• The Egyptian Rhinnd Papyrus is dated about 1650BC and has $4(8/9)^2 = 3.16$ as an estimate (2, pg 1)
• Archimedes of Syracuse obtained the first theoretical approximation of $\frac{223}{71} < \pi < \frac{22}{7}$ (2, pg 1)
• Archimedes used trig for his estimates, which was unhistorical (2, pg 2)
• Some others to give estimates were
  - Ptolemy about 150AD to 3 places
  - ZuChongzhi about 500AD as 355/113
  - alKhwarizmi about 800 to 3 places
  - al Kashi in 1430 to 14 places
  - Viete in late 1500s to 9 places
  - Roomen in 1600s to 17 places
  - Van Ceulen in 1600 to 35 places (2, pg 2)
• Wallis (1616-1703) developed mathematical formulae for pi (2, pg 2)
• Gregory and Leibniz also developed formulas (2, pg 2)
• Other more refined estimates are:
  1699 Sharp used Gregory's to get 71 digits
  1701 Machin improved the method and got 100 digits
  1719 deLangy used Machin's to get 112
  1789 Vega refined and got 126 and later 136 in 1794
  1841 Rutherford got 152 and later 440 in 1853
  1873 Shanks got to 527 correctly (2, pg 3)
  1949 a computer calculated it to 2000 places

• The first to use the symbol π with its present meaning was a Welsh named Jones in 1706.
  Euler adopted it in 1737 and it became standard (2, pg 4)

1. http://www.scit.wlv.ac.uk/university/scit/modules/mm2217/atan.htm
2. http://www-gap.dcs.st-and.ac.uk/~history/HistTopics/Pi_through_the_ages.html