Topic 4 – Sequences
d) Cauchy (3.6) and Limits at Infinity (3.7) Handout Notes
Assigned Problems: pg 60 (1,2,4 and 5) and pg 64 (1-3, 5)

- A sequence \((x_n)_{n \in \mathbb{N}}\) is **Cauchy** if \(\forall \varepsilon > 0 \ \exists \ n_0 \in \mathbb{N} \text{ s.t. for } n, m \geq n_0 \ |x_n - x_m| < \varepsilon\)

**Proposition 3.5** A convergent sequence is Cauchy

**Theorem** A Cauchy sequence is bounded

**Proof:**
Let \((x_n)_{n \in \mathbb{N}}\) be Cauchy

Then \(\forall \varepsilon > 0 \ \exists \ n_0 \in \mathbb{N} \text{ s.t. for } n, m \geq n_0 \ |x_n - x_m| < \varepsilon\)

Let \(\varepsilon = 1\) and \(m = n_0\)

Then \(|x_n - x_{n_0}| < 1\)

By the triangle inequality, \(|x_n - x_{n_0}| \geq |x_n| - |x_{n_0}|\)

Therefore, \(-1 + |x_{n_0}| < |x_n| < 1 + |x_{n_0}|\)

This implies that \(|x_n| < 1 + |x_{n_0}| \ \forall n \geq n_0\)

For those values in the sequence that are less than \(n_0\), let \(B = \max \{|x_1|, |x_2|, |x_3|, \ldots, |x_{n_0-1}|, 1+|x_{n_0}|\}\)

Then \(|x_n| < B \ \forall n\). This implies that the sequence \((x_n)_{n \in \mathbb{N}}\) is bounded

**Theorem 3.12** A sequence in \(\mathbb{R}^e\) is Cauchy if and only if it converges

**Proof:**
Convergent implies Cauchy follows directly from proposition 3.5

Cauchy implies convergent:

Let \((x_n)_{n \in \mathbb{N}}\) be Cauchy

Then Lemma 3.3 says that \((x_n)_{n \in \mathbb{N}}\) is bounded

Then by the BW theorem, \((x_n)_{n \in \mathbb{N}}\) has a convergent subsequence

Let \((x_{n_k})_{k=1}^\infty\) be the sequence where \(x_{n_k} \rightarrow x\)

We must show that \(x_n \rightarrow x\)

\((x_n)_{n \in \mathbb{N}}\) Cauchy implies \(\forall \varepsilon > 0 \ \exists N_0 \in \mathbb{N} \text{ s.t. for } n, m \geq N_0 \ |x_n - x_m| < \frac{\varepsilon}{2}\)

\(x_{n_k} \rightarrow x\) implies that \(\exists \varepsilon > 0 \ \exists N_1 \in \mathbb{N} \text{ s.t. for } k \geq N_1 \ |x_{n_k} - x| < \frac{\varepsilon}{2}\)

\(|x_n - x| = |x_n - x_{n_k} + x_{n_k} - x| \leq |x_n - x_{n_k}| + |x_{n_k} - x|\)

Now if \(n, n_k \geq N_0\) and \(k \geq N_1\) \(\Rightarrow |x_n - x| < \varepsilon\)

So if we let \(n_0 = \max \{N_0, N_1\}\), then \((x_n)_{n \in \mathbb{N}}\) converges to \(x\)
• If \((x_n)_{n \in \mathbb{N}}\) has a limit of positive or negative infinity, then it **diverges**.

• \(x_n \to \infty\) iff \(\forall \alpha > 0 \exists n_0 \in \mathbb{N} s.t. x_n > \alpha \forall n \geq n_0\)

• \(x_n \to -\infty\) iff \(\forall \beta < 0 \exists n_0 \in \mathbb{N} s.t. x_n < \beta \forall n \geq n_0\)

• **Theorem 3.14** For \(x_n \to x, y_n \to \infty, z_n \to -\infty\)
  if \(-\infty < x \leq \infty\), \(x_n + y_n \to \infty\)
  if \(-\infty \leq x < \infty\), \(x_n + z_n \to -\infty\)
  if \(0 < x \leq \infty\), \(x_n \cdot y_n \to \infty, x_n \cdot z_n \to -\infty\)
  if \(-\infty \leq x < \infty\), \(x_n \cdot y_n \to -\infty, x_n \cdot z_n \to \infty\)
  if \(-\infty < x < \infty\), \(\frac{x_n}{y_n} \to 0, \frac{y_n}{z_n} \to 0\)

• All other theorems are repeated in this section with positive/negative infinity included as possible limit values.

• **For example:** \(x_n \to x\) \((x \in \mathbb{R}^\#)\), then every subsequence converges to \(x\)

• **Proposition 3.6**

\[
\lim_{n \to \infty} x_n \neq \infty \iff (x_n)_{n \in \mathbb{N}} \text{ has a subsequence that is bounded above}
\]

\[
\lim_{n \to -\infty} x_n \neq -\infty \iff (x_n)_{n \in \mathbb{N}} \text{ has a subsequence that is bounded below}
\]

• **Theorem 3.16** If \((x_n)_{n \in \mathbb{N}}\) is a monotone sequence in \(\mathbb{R}\), then it has a limit in the extended real number system