c) Monotone Sequences (3.4) and Bolzano-Wierstrass Handout (3.5) Notes

Assigned Problems: Page 54 (1-7), pg 57 (1-3, 5-8)

- A sequence \((x_n)\) in \(\mathbb{R}\) is **monotone increasing** if \(x_n \leq x_{n+1} \ \forall n \in N\)
- A sequence \((x_n)\) in \(\mathbb{R}\) is **strictly increasing** if \(x_n < x_{n+1} \ \forall n \in N\)

- A sequence \((x_n)\) in \(\mathbb{R}\) is **monotone decreasing** if \(x_n \leq x_{n+1} \ \forall n \in N\)
- A sequence \((x_n)\) in \(\mathbb{R}\) is **strictly decreasing** if \(x_n > x_{n+1} \ \forall n \in N\)

- A sequence is **monotone** if it is either monotone increasing or monotone decreasing
- Can a sequence be both monotone increasing and monotone decreasing?
- Can a sequence be both strictly increasing and strictly decreasing?

**Theorem 3.7** A bounded monotone sequence converges

*Proof:*

Suppose \((x_n)\) is a bounded monotone increasing sequence

Then, \((x_n)\) has a “largest value” called a supremum by the completeness axiom

Let \(\alpha = \sup \{x_n \ \forall n \in N\}\)

We need to show that \((x_n)\) converges to \(\alpha\)

Let \(\varepsilon > 0\)

By proposition 2.5 there is a \(n_0 \in N\) such that \(\alpha - \varepsilon < x_{n_0}\)

\((x_n)\) is monotone increasing implies that \(x_{n_0} < x_n\) for \(n \geq n_0\)

But \(\alpha\) is the sup., so \(x_n \leq \alpha\) \(\forall n\)

Therefore, \(\forall n \geq n_0\) \(\alpha - \varepsilon < x_n \leq \alpha\), which implies that \(0 < \alpha - x_n < \varepsilon\)

Therefore, \(|x_n - \alpha| < \varepsilon\) \(\forall n \geq n_0\)

Suppose \((x_n)\) is a bounded monotone decreasing sequence...

- For \(A_n\) a subset of \(\mathbb{R}\), \((A_n)_{n \in N}\) is a **sequence of nested subsets** of \(\mathbb{R}\) which is **nested upward** if \(A_n \subset A_{n+1} \ \forall n \in N\) and **nested downward** if \(A_n \supset A_{n+1} \ \forall n \in N\)

**Example:** The sequence defined by \(\left[0, \frac{1}{n}\right]_{n \in N}\) is equivalent to \([0,1], [0, \frac{1}{2}], [0, \frac{1}{3}]\ldots\). Is this nested up or down?

**Theorem 3.9** Every sequence in \(\mathbb{R}\) has a monotone subsequence
Theorem 3.10 (BW Theorem for sequences)  A bounded sequence in Re has a convergent subsequence

Proof:
Let \((x_n)\) be a bounded sequence in Re
By Thm 3.9 (pg. 53), every sequence in Re has a montone subsequence
So \((x_n)\) has a monotone subsequence
Since \((x_n)\) is bounded, this subsequence is bounded
By Thm 3.7 (pg. 51), a bounded monotone sequence converges, so this subsequence converges

Let \(A\) be a subset of Re and let \(x\) be in Re. Then \(x\) is an accumulation point of \(A\) if every neighborhood of \(x\) contains a point of \(A\) (not \(x\)).

If \(x\) is in \(A\) and \(x\) is not an accumulation point, it is an isolated point

Example:
Let \(A = [-10, 0] \cup \{2\}\)
Accumulation points are \([-10, 0]\), and 2 is an isolated point

Proposition 3.4 If \(A\) is a subset of Re and \(x\) is in Re, \(x\) is an accumulation point of \(A\) if and only if every neighborhood of \(x\) contains infinitely many points in \(A\).

Proof:
Let \(x\) be an accumulation point of \(A\) and let \(U\) be a neighborhood of \(x\)
Then \(U = \{y \mid |x - y| < \varepsilon\}\)
Suppose \(U\) has a finite number of points of \(A\)
Then \((U - \{x\}) \cap A = \{x_1, x_2, \ldots, x_n\}\)
Let \(\varepsilon = \min \{|x - x_1|, |x - x_2|, \ldots, |x - x_n|\}\)
Then the neighborhood \(\left(x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2}\right)\) contains no points of \(A\)
This contradicts \(x\) being an accumulation point.

Theorem 3.11 Every bounded infinite subset of Re has an accumulation point in Re