Definitions:
- A function $f$ is **differentiable** at $c$ if and only if
  \[ \exists L \in \mathbb{R} \text{ s.t. } \forall \epsilon > 0 \exists \delta > 0 \text{ s.t. for } x \in I \text{ and } 0 < |x - c| < \delta \text{ then } \left| \frac{f(x) - f(c)}{x - c} - L \right| < \epsilon. \]
- The derivative is infinite, i.e. $f'(c) = \infty$ if and only if
  \[ \forall \alpha > 0 \exists \delta > 0 \text{ s.t. for } x \in I \text{ and } 0 < |x - c| < \delta \text{ then } \frac{f(x) - f(c)}{x - c} > \alpha. \]
- And the derivative is negatively infinite, i.e. $f'(c) = -\infty$ if and only if
  \[ \forall \beta < 0 \exists \delta > 0 \text{ s.t. for } x \in I \text{ and } 0 < |x - c| < \delta \text{ then } \frac{f(x) - f(c)}{x - c} < \beta. \]
- NOTE: The derivative can exist in the extended real number system, which means if it is positive or negative infinity, than the slope of the tangent line is vertical
- If $f'(c) \in \mathbb{R}$ then $f$ is **differentiable at $c$**. If $f$ is differentiable for all values in some domain $I$, then it is **differentiable on $I$**. If $f$ is differentiable for all values in its entire domain, than it is said to be **differentiable**.

**Differentiable Implies Continuous (Proof):**
Continuous says: \[ \forall \epsilon > 0 \exists \delta > 0 \text{ s.t. if } |x - c| < \delta \text{ then } |f(x) - f(c)| < \epsilon. \]
Differentiable says: \[ \forall \epsilon > 0 \exists \delta > 0 \text{ s.t. for } |x - c| < \delta \text{ then } \left| \frac{f(x) - f(c)}{x - c} - L \right| < \epsilon, \text{ and } \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = L \in \mathbb{R}. \]
Let $\delta, \epsilon > 0$ and $|x - c| < \delta$ for $x \in D$.
Then, $f$ differentiable implies that \[ \left| \frac{f(x) - f(c)}{x - c} - L \right| < \epsilon \]
By the Triangle Inequality, \[ \left| \frac{f(x) - f(c)}{x - c} - L \right| \geq \left| \frac{f(x) - f(c)}{x - c} \right| - |L| \]
So \[ -\epsilon < -\epsilon \left| \frac{f(x) - f(c)}{x - c} \right| - |L| < \epsilon, \text{ and } |f(x) - f(c)| < (\epsilon + |L|) \cdot |x - c| \]
Therefore, \[ |f(x) - f(c)| < \delta \cdot (\epsilon + |L|). \]
Let \[ \delta = \frac{\epsilon}{\epsilon + |L|}. \]
Then \[ \forall \epsilon > 0 \exists \delta = \frac{\epsilon}{\epsilon + |L|} > 0 \text{ s.t. for } |x - c| < \delta \text{ then } |f(x) - f(c)| < \epsilon. \]
Other Ideas:
- Continuous does not imply differentiable. **How can we prove this?**
- Proposition: If \( f \) and \( g \) are differentiable at \( c \), then \( f \cdot g, f + g, a \cdot f \) (for constant \( a \)) and \( f / g \) (\( g(c) \neq 0 \)) are all differentiable.
- Theorem: Let \( f : I \rightarrow J \) be differentiable at \( c \), and \( g : J \rightarrow \mathbb{R} \) be differentiable at \( f(c) \). Then \( g[f(x)] \) is differentiable at \( c \), and \( g[f(x)]' = g'(f(c)) \cdot f'(c) \).
- If \( f : I \rightarrow \mathbb{R} \) with \( c \in I \). \( f \) has a **local max** at \( c \) if there is a neighborhood \( U \) of \( I \) such that \( f(x) \leq f(c) \ \forall x \in U \cap I \).
- If \( f : I \rightarrow \mathbb{R} \) with \( c \in I \). \( f \) has a **local min** at \( c \) if there is a neighborhood \( U \) of \( I \) such that \( f(x) \geq f(c) \ \forall x \in U \land I \).

**Proposition:**
Suppose \( f \) has a local max or min at an interior point \( c \in I \). If \( f \) is differentiable at \( c \), then \( f'(c) = 0 \).

Proof (with max):
Let \( f \) have a local max at an interior point \( c \).
Then there is a neighborhood \( U \) such that \( f(x) \leq f(c) \ \forall x \in U \land I \).
Therefore, \( \exists \delta > 0 \) s.t. \( f(x) \leq f(c) \ \forall x \in (c - \delta, c + \delta) \).
Therefore, \( c - \delta < x < c + \delta \).
For \( x \in (c, c + \delta) \):
\[ x - c > 0 \text{ and } f(x) - f(c) \leq 0 \]
Therefore, \( \frac{f(x) - f(c)}{x - c} \leq 0 \)
Since \( f \) is differentiable, \( f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \) exists.
An earlier proposition implies that for \( f(x) \in [a, b] \), then \( \lim_{x \to c} f(x) \in [a, b] \).
Therefore, \( f'(c) \leq 0 \)
For \( x \in (c - \delta, c) \):
\[ x - c < 0 \text{ and } f(x) - f(c) \leq 0 \]
Therefore, \( \frac{f(x) - f(c)}{x - c} \geq 0 \)
Since \( f \) is differentiable, \( f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \) exists.
An earlier proposition implies that for \( f(x) \in [a, b] \), then \( \lim_{x \to c} f(x) \in [a, b] \).
Therefore, \( f'(c) \geq 0 \)
Since this must work \( \forall x \in (c - \delta, c + \delta) \), \( f'(c) = 0 \)

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