Topic 3 – Derivative

a) Differentiation Basics

Assigned Problems: “Finney” pg 117 (1-50)

The Background:
- From an original starting point \((a, f(a))\) we are looking at the slope of the line connecting it to another point a distance of \(h\) away from \(a\).

![Graph showing the slope of a line connecting points \((a, f(a))\) and \((a+h, f(a+h))\).]

- What is the slope of this dashed line connecting \((a, f(a))\) and \((a+h, f(a+h))\)?
  \[
  \Delta y = f(a+h) - f(a) \\
  \Delta x = (a+h) - a = h \\
  \text{slope} = \frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{h}
  \]

- We want to let \(h\) tend to zero, so this point collapses into \((a, f(a))\), so we take \(\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}\).

- This is slope of the tangent line, also the instantaneous rate of change of \(f(x)\) at \(x = a\), also called the derivative of \(f(x)\) at \(x = a\).

Some Examples:
- Find the slope of the tangent line to the parabola \(y = x^2 - 3x\) at the point \(x = -1\)
  
  First, \(y(-1) = 4\)
  
  \[
  \lim_{h \to 0} \frac{y(-1+h) - y(-1)}{h} = \lim_{h \to 0} \frac{((-1+h)^2 - 3(-1+h)) - 4}{h} = \lim_{h \to 0} \frac{1 - 2h + h^2 + 3 + 3h - 4}{h}
  \]
  
  \[
  = \lim_{h \to 0} \frac{-5h + h^2}{h} = \lim_{h \to 0} (-5 + h) = -5
  \]
• Find an equation of the tangent line to the curve \( y = \sqrt{5x - 4} \) at the point \( x = 4 \).

\[
y(4) = \sqrt{5(4) - 4} = 4
\]

\[
\lim_{{h \to 0}} \frac{y(4+h) - y(4)}{h} = \lim_{{h \to 0}} \frac{\sqrt{5(4+h) - 4} - 4}{h} = \lim_{{h \to 0}} \frac{\sqrt{16+5h} - 4}{\sqrt{16+5h} + 4} = \frac{5}{8}
\]

The line has a slope of 5/8 and goes through the point (4,4)

\[
y - 4 = \frac{5}{8}(x - 4)
\]

• Find the slope of the tangent to the curve \( y = \frac{1}{x-3} \) at any point \( a \)

\[
\lim_{{h \to 0}} \frac{y(a+h) - y(a)}{h} = \lim_{{h \to 0}} \frac{\frac{1}{a+h-3} - \frac{1}{a-3}}{h} = \lim_{{h \to 0}} \frac{a-3}{h(a+h-3)(a-3)} - \frac{a+h-3}{h(a+h-3)(a-3)}
\]

\[
= \lim_{{h \to 0}} \frac{a-3 - (a+h-3)}{h(a+h-3)(a-3)} = \frac{-h}{h(a+h-3)(a-3)} = \frac{-1}{(a-3)^2}
\]

• Find the slope of the tangent to the curve \( y = 2\sqrt{x} \).

This time we will leave it as \( x \) instead of \( a \)

\[
\lim_{{h \to 0}} \frac{y(x+h) - y(x)}{h} = \lim_{{h \to 0}} \frac{2\sqrt{x+h} - 2\sqrt{x}}{h} = \lim_{{h \to 0}} \frac{2\sqrt{x+h} - 2\sqrt{x}}{2\sqrt{x+h} + 2\sqrt{x}} = \frac{4}{4\sqrt{x}} = \frac{1}{\sqrt{x}}
\]

Definitions:

• The **derivative of a function** \( f(x) \) in general is \( \lim_{{h \to 0}} \frac{f(x+h) - f(x)}{h} \) provided the limit exists.

• There are many symbols used to describe the derivative of \( f(x) \). Remember they are all operators

\[
f'(x) = \frac{df}{dx} = \frac{d}{dx} f(x)
\]

• Remember, this measures the slope of the line that is tangent to \( f' \) at any particular point. It is also called the instantaneous rate of change

• If the limit exists at \( x = a \), then the function is said to be **differentiable** at \( x = a \)

• If the limit exists on some interval (specifically, for all values within that interval) then the function is said to be **differentiable on that interval**

• If the limit exists for all values, then the function is said to be **differentiable**. This is the same concept as continuous, where if no point or interval is given, you assume it means everywhere.

• So long as the derivative exists, it can be interpreted as a function itself, and it is useful to help us understand the behavior of \( f(x) \)
Critical Points:

- What do you notice about the tangent lines given in the following picture?

What is so special about these points?

- The above points are examples of critical points, or points where the derivative is zero or undefined.
- You can ‘match’ a graph of a function with its derivative by looking at the critical points, whenever you have a critical point on your original function, the derivative function will be zero there.
- Can you identify which is the original function and which is the derivative?
• Can you identify which is the original function and which is the derivative?

• When a function is increasing, the slope of the tangent lines will be positive, hence the derivative is positive when a function is increasing.
• In a similar manner, the derivative is negative when a function is decreasing.
• When a function changes from increasing to decreasing, you have a critical point, where the derivative goes to zero.