A First Look, When Things Go Right:

- From what we saw from the last section, does the limit of \( f(x) \) exist at \( x = a \) for the graph below?

- Let’s pick two horizontal lines that are the same distance from \( f(a) \), which is the limit, and plot them on our graph. We can make them as close or as far from \( f(a) \) as we want, it is our choice.

- Now we have to make a rectangle with two vertical lines that are the same distance from \( a \), but the trick is that we have to make sure our function only enters and exits our rectangle from the sides, not the top or bottom.

- No matter how small we make our horizontal lines, we can find vertical lines to make it work.
A First Look, When Things Go Wrong:

- From what we saw from the last section, does the limit of \( f(x) \) exist at \( x = a \) for the graph below?

- Well, it looks like for the horizontal lines I’ve chosen, I can get a box around them

- But remember, I can pick *any* horizontal lines I want and it still has to work

- This is bad news. There is no way I can even get my function inside the box at all, let alone enter and exit from the sides.
- The limit here does not exist at \( a \)
Translating the Pictures to a Definition:

- Let’s look at a specific function, \( f(x) = x + 1 \)
- This is a linear function, so it has a limit for every domain value, but let’s pick \( a = 2 \)
- Now we’ve already learned that \( \lim_{x \to 2} x + 1 = 3 \), but now we want to formally prove it is true
- The horizontal lines we pick out of the blue are centered about the limit \( L \), in this case 3

\[
\begin{align*}
\text{(2,3)} & \quad 3 + \varepsilon \\
3 \quad & \quad 3 - \varepsilon \\
\hline
2
\end{align*}
\]

We require our function to be inside these lines, that is, \( f(x) < 3 + \varepsilon \) and \( f(x) > 3 - \varepsilon \)

We can say this with \( |f(x) - 3| < \varepsilon \)

So a way to specify the horizontal lines in general would be: \( |f(x) - L| < \varepsilon \) for any \( \varepsilon > 0 \)

- Now for the vertical lines. Remember, our function has to go in and out the sides.

\[
\begin{align*}
\text{(2,3)} & \quad 2 - \delta \quad 2 + \delta \\
2 \quad & \quad 2
\end{align*}
\]

We found two red lines that will work for any \( x \) between them, that is \( x > 2 - \delta \) and \( x < 2 + \delta \)

We can say this with \( |x-2| < \delta \)

So a way to specify the vertical lines in general would be to say \( |x-a| < \delta \)

- So remember our game… for any \( \varepsilon > 0 \) that is chosen, we have to find a \( \delta > 0 \) so that as long as \( |x-a| < \delta \) we satisfy \( |f(x) - L| < \varepsilon \)

There you have it! The formal definition of a limit…

The limit of \( f(x) \) as \( x \) approaches \( a \) is \( L \), i.e. \( \lim_{x \to a} f(x) = L \) if and only if for all \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that \( |f(x) - L| < \varepsilon \) whenever \( |x-a| < \delta \).
**Another Small Step:**

- **Use the graph to find delta so** \( |f(x) - 4| < 0.1 \) whenever \( |x - 2| < \delta \)

In the box you are looking at, the function is touching the corners. Plus, it is not symmetric about \( x = 2 \).
Remember it needs to be the same distance on the right and left of 2.
The distance from 1.5 to 2 is 0.5, and the distance from 2 to 2.8 is 0.8.
We need to take the smaller of these.
So any positive value of delta less than (or equal to) 0.5 will work. i.e. 0.5, 0.49, 0.4

- **For** \( f(x) = x^2 \), **find a number delta >0 such that** \( |x^2 - 4| < 0.5 \) whenever \( |x - 2| < \delta \)

In this example, we are trying to “prove” \( \lim_{x \to 2} x^2 = 4 \) with a specific epsilon = 0.5
\[ |x^2 - 4| < 0.5 \Rightarrow x^2 > 3.5 \quad \text{and} \quad x^2 < 4.5 \]
These are the top and bottom lines of your box.
You need to find out what input gives these two outputs. i.e. \( f^{-1}(3.5) \) and \( f^{-1}(4.5) \)
\( f^{-1}(x) = \sqrt{x} \), so \( f^{-1}(3.5) \approx 1.87 \) and \( f^{-1}(4.5) \approx 2.12 \)
We pick the smallest of these distances from 2, so delta = 0.12 (or less)

**A Practical Example:**

- **Prove that** \( \lim_{x \to 3} (2x - 5) = 1 \)
- Here, \( f(x) = 2x - 5 \)
  \( a = 3 \)
  \( L = 1 \)
- Let \( \varepsilon > 0 \) be given and assume that \( |x - 3| < \delta \) for some \( \delta > 0 \). Remember, it is our job to find that delta!
- \( |f(x) - L| = |2x - 5 - 1| = |2x - 6| = 2|x - 3| < 2\delta \)
- We want to force \( |f(x) - L| < \varepsilon \), so we want \( 2\delta < \varepsilon \) or \( \delta < \frac{\varepsilon}{2} \).
- So this means that, \( \lim_{x \to 3} (2x - 5) = 1 \) because for all \( \varepsilon > 0 \) there is a \( \delta > 0 \) (i.e. \( \delta < \frac{\varepsilon}{2} \)) such that \( |f(x) - L| < \varepsilon \) whenever \( |x - 3| < \delta \).
Another Example:

- Prove that \( \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2 \)

- First note that for \( x \neq 1 \), \( \frac{x^2 - 1}{x - 1} = x + 1 \)

- Let \( \varepsilon > 0 \) be given and assume that \( |x - 1| < \delta \) for some \( \delta > 0 \). This means that \( x \neq 1 \)

- \( |f(x) - L| = |x + 1 - 2| = |x - 1| < \delta \)

- We want to force \( |f(x) - L| < \varepsilon \), so we want \( \delta < \varepsilon \).

- So this means that, \( \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2 \) because for all \( \varepsilon > 0 \) there is a \( \delta > 0 \) (i.e. \( \delta < \varepsilon \)) such that \( |f(x) - 2| < \varepsilon \) whenever \( |x - 1| < \delta \).
Another Example:

- **Prove that** \( \lim_{x \to 1} \sqrt{10 - x} = 3 \)
- Let \( \epsilon > 0 \) be given and assume that \( |x - 1| < \delta \) for some \( \delta > 0 \).
- \[ |f(x) - L| = \left| \sqrt{10 - x} - 3 \right| < \epsilon \]
  \[ -\epsilon + 3 < \sqrt{10 - x} < \epsilon + 3 \]
  \[ (3 - \epsilon)^2 < 10 - x < (\epsilon + 3)^2 \quad \text{here we must ensure that} \quad \epsilon < 3 \]
  \[ (3 - \epsilon)^2 - 10 < -x < (\epsilon + 3)^2 - 10 \]
  \[ 10 - (\epsilon + 3)^2 < x < 10 - (3 - \epsilon)^2 \]
  \[ 1 - \epsilon^2 - 6\epsilon < x < 1 - \epsilon^2 + 6\epsilon \]
  \[ -\epsilon^2 - 6\epsilon < x - 1 < -\epsilon^2 + 6\epsilon \]
- We want to force \( |f(x) - L| < \epsilon \), so we want \( \delta = \min(-\epsilon^2 - 6\epsilon, -\epsilon^2 + 6\epsilon) \).
- Note that the minimum value is actually \( -\epsilon^2 + 6\epsilon \).

Or another way, note that for any given \( \epsilon > 0 \), we must ensure that \( \delta > 0 \). Notice that we have the following...

<table>
<thead>
<tr>
<th>( \epsilon )</th>
<th>( -\epsilon^2 - 6\epsilon )</th>
<th>( -\epsilon^2 + 6\epsilon )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.01</td>
<td>0.0001</td>
<td>0.0001</td>
</tr>
<tr>
<td>0.02</td>
<td>0.0404</td>
<td>0.0404</td>
</tr>
<tr>
<td>0.03</td>
<td>0.0909</td>
<td>0.0909</td>
</tr>
<tr>
<td>0.04</td>
<td>0.1616</td>
<td>0.1616</td>
</tr>
<tr>
<td>0.05</td>
<td>0.2525</td>
<td>0.2525</td>
</tr>
<tr>
<td>0.06</td>
<td>0.3636</td>
<td>0.3636</td>
</tr>
<tr>
<td>0.07</td>
<td>0.4949</td>
<td>0.4949</td>
</tr>
<tr>
<td>0.08</td>
<td>0.6464</td>
<td>0.6464</td>
</tr>
<tr>
<td>0.09</td>
<td>0.8182</td>
<td>0.8182</td>
</tr>
<tr>
<td>0.1</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

So this means that \( \lim_{x \to 1} \sqrt{10 - x} = 3 \) because for all \( \epsilon > 0 \) there is a \( \delta > 0 \) (i.e. \( \delta = \min(-\epsilon^2 - 6\epsilon, -\epsilon^2 + 6\epsilon) \)) such that \( |f(x) - 3| < \epsilon \) whenever \( |x - 1| < \delta \).
Another Example:

- Prove that \( \lim_{x \to 2} \frac{1}{x^2} = \frac{1}{4} \)

- Let \( \varepsilon > 0 \) be given and assume that \( |x - 2| < \delta \) for some \( \delta > 0 \)

\[
|f(x) - L| = \left| \frac{1}{x^2} - \frac{1}{4} \right| < \varepsilon
\]

- \( -\varepsilon + \frac{1}{4} < \frac{1}{x^2} < \varepsilon + \frac{1}{4} \)

- \(\frac{1}{\varepsilon + \frac{1}{4}} < x^2 < \frac{1}{\varepsilon + \frac{1}{4}} \)

\[
\sqrt{\frac{4}{4\varepsilon + 1}} < x < \sqrt{\frac{4}{-4\varepsilon + 1}}
\]

- We need to assure that \(-4\varepsilon + 1 > 0, \varepsilon < \frac{1}{4} \)

\[
\sqrt{\frac{4}{4\varepsilon + 1}} - 2 < x - 2 < \sqrt{\frac{4}{-4\varepsilon + 1}} - 2
\]

- We want to force \( |f(x) - L| < \varepsilon \), so we want \( \delta = \min \left( \sqrt{\frac{4}{4\varepsilon + 1}} - 2, \sqrt{\frac{4}{-4\varepsilon + 1}} - 2 \right) \)

- If we want to investigate further, we can find the minimum...

\[
\delta > 0 \text{ i.e. } \delta = \min \left( \sqrt{\frac{4}{4\varepsilon + 1}} - 2, \sqrt{\frac{4}{-4\varepsilon + 1}} - 2 \right)
\]

such that \( |f(x) - \frac{1}{4}| < \varepsilon \) when \( |x - 2| < \delta \).
**More on Limits:**

- **Infinite Limits:**
  \[
  \lim_{x \to a} f(x) = \infty \iff \forall B > 0 \; \exists \delta > 0 \text{ s.t. } |x - a| < \delta \Rightarrow f(x) > B
  \]
  \[
  \lim_{x \to a} f(x) = -\infty \iff \forall B > 0 \; \exists \delta > 0 \text{ s.t. } |x - a| < \delta \Rightarrow f(x) \leq -B
  \]

- **Using subsequences:**
  \[
  \lim_{x \to c} f(x) = L \iff \lim_{n \to \infty} f(x_n) = L \ \forall (x_n) \text{ in } D \text{ s.t. } x_n \neq c \forall n \text{ and } \lim x_n = c.
  \]

- **Example:** Show \( f(x) = \begin{cases} 0 & x \in Q \\ 1 & x \in \mathbb{R} - Q \end{cases} \) does not have a limit at any point.

  Let \( c \in \mathbb{R} \) be such that \( f(x) \) has a limit at \( c \)
  
  Let \( x_n \) be a sequence with \( x_n \in Q - \{c\} \) with \( x_n \to c \).
  
  Then \( x_n \to c \) and \( f(x_n) \to 0 \)
  
  Let \( y_n \) be a sequence with \( y_n \in \mathbb{R} - Q - \{c\} \) with \( y_n \to c \).
  
  Then \( y_n \to c \) and \( f(y_n) \to 1 \)
  
  This is a contradiction to the above proposition.