Section 4.5 – Eigenvectors and Eigenspaces

Homework (page 314) problems 1-17

Geometric Multiplicity:

- Find the eigenvectors and eigenvalues for the matrix \( A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \).

Solution: \( |A - \lambda I| = \begin{vmatrix} 1 - \lambda & 1 & 0 \\ 0 & 1 - \lambda & 1 \\ 0 & 0 & 1 - \lambda \end{vmatrix} = (1 - \lambda)^3 \). So \( \lambda = 1 \) is an eigenvalue (multiplicity 3).

The eigenvector corresponding to it is \( A - I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow x_1 = 0, x_3 = 0 \). \( \vec{x} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \).

- For an \( nxn \) matrix \( A \), if \( \lambda \) is an eigenvalue, then
  - The null space of \( A - \lambda I \) is denoted by \( E_\lambda \) and is called the eigenspace of \( \lambda \).
  - The dimension of \( E_\lambda \) is called the geometric multiplicity of \( \lambda \).

So in the example above, \( E_1 = \left\{ \vec{x} : \vec{x} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, a \neq 0 \right\} \), and its dimension is 1, so \( \lambda = 1 \) has geometric multiplicity of 1, and algebraic multiplicity of 3.

- When an \( nxn \) matrix has a eigenvalue whose geometric multiplicity is less than the algebraic multiplicity, then it is called a defective matrix. In essence, it will not have a set of \( n \) linearly independent eigenvectors.

- For a set of \( k \) eigenvectors corresponding to \( k \) distinct eigenvalues of an \( nxn \) matrix \( A \), then these eigenvectors form a linearly independent set (Theorem 15).

- For an \( nxn \) matrix, if it has \( n \) distinct eigenvalues, then it has a set of \( n \) linearly independent eigenvectors (Corollary).
Exercise 12. Find the eigenvectors/values for $A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$. Is $A$ defective?

Solution: $|A - \lambda I| = \begin{vmatrix} 1 - \lambda & 1 & -1 \\ 0 & 2 - \lambda & -1 \\ 0 & 0 & 1 - \lambda \end{vmatrix} = (1 - \lambda)(2 - \lambda)(1 - \lambda)$

So $\lambda_1 = 1$ (algebraic multiplicity 2), $\lambda_2 = 2$ (algebraic multiplicity 1)

For $1$:

$$\begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow x_2 = x_3, \quad \tilde{x} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ (geom. multiplicity 2)

For $2$:

$$\begin{bmatrix} -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \Rightarrow x_3 = 0, \quad x_1 = x_2; \quad \tilde{x} = x_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$ (geom. mult. 1)

This is not a defective matrix.