Section 3.2 – Vector Space Properties of $\mathbb{R}^n$

Homework (pages 174-175) problems 1-18

Properties in $\mathbb{R}^n$:

- Recall a vector in $n$-dimensional space will have the form $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$.
- The following hold for $n$-dimensional vector spaces (Theorem 1):
  - Closure under addition and scalar multiplication
  - Commutative and associative properties of addition
  - Additive identity, namely $\vec{0}$
  - Additive inverse exists, namely $-\vec{x}$
  - Properties of scalar multiplication, i.e.
    \[
    a(b\vec{x}) = (ab)\vec{x}, \quad a(\vec{x} + \vec{y}) = a\vec{x} + ay, \quad (a + b)\vec{x} = a\vec{x} + b\vec{x}, \quad 1\vec{x} = \vec{x}
    \]
- A subset that satisfies all the properties of Theorem 1 is called a subspace of $\mathbb{R}^n$.
- A subset, $W$, of $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ iff
  - It contains the zero vector
  - For any two vectors in $W$, their sum is in $W$
  - For any vector $\vec{x}$ in $W$ and any scalar $a$, $a\vec{x} \in W$
- Exercise 2. Determine if $W = \{ \vec{x} | x_1 - x_2 = 2 \}$ is a subspace of $\mathbb{R}^2$.

Solution:

The vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ does not satisfy $x_1 - x_2 = 2$ (so it is not a subspace)

Also, $\vec{x} + \vec{y}$ is not in the space. Given $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, with $x_1 = 2 + x_2$, $y_1 = 2 + y_2$,

$\vec{x} + \vec{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}$. But from above, $x_1 + y_1 = 4 + 2x_2 + 2y_2 \neq 2 + (x_2 + y_2)$

Also, $a\vec{x}$ is not in the space. Given $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ with $x_1 = 2 + x_2$, $a\vec{x} = \begin{bmatrix} ax_1 \\ ax_2 \end{bmatrix}$,

and from above, $ax_1 = 2a + ax_2 = a(2 + x_2) \neq 2 + ax_2$.

So for all these reasons (only one needs to fail) $W$ is not a subspace of $\mathbb{R}^2$. 
• Determine if \( W = \{ \vec{x} \mid \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, x_1 = x_2 - x_3, \ x_2, x_3 \in \mathbb{R} \} \) is a subspace of \( \mathbb{R}^3 \).

Solution:

The vector \( \vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \) satisfies \( x_1 = x_2 - x_3 \)

For any vectors \( \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \) and \( \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \) with \( x_1 = x_2 - x_3 \) and \( y_1 = y_2 - y_3 \),

\[ \vec{x} + \vec{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}. \] From above, \( x_1 + y_1 = (x_2 - x_3) + (y_2 - y_3) = (x_2 + y_2) - (x_3 + y_3) \)

For any scalar \( a \), \( a\vec{x} \) is in the space. Given \( \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \) with \( x_1 = x_2 - x_3 \), \( a\vec{x} = \begin{bmatrix} ax_1 \\ ax_2 \\ ax_3 \end{bmatrix} \),

and from above, \( ax_1 = ax_2 - ax_3 = a(x_2 - x_3) \).

Therefore, \( W \) is a subspace, and it represents the plane with equation \( x - y + z = 0 \).