Section 1.9 – Matrix Inverses and Their Properties
Homework (pages 102-103) problems 1-8, 13-28

The Matrix Inverse:
• The inverse of an \( nxn \) matrix \( A \), denoted \( A^{-1} \), satisfies the system \( A^{-1}A = AA^{-1} = I \) (where \( I \) is the identity matrix).
• If we can find an inverse of \( A \) (i.e. if the inverse exists) then we say that \( A \) is invertible.
• \( A \) has an inverse if and only if \( A \) is nonsingular (recall nonsingular iff linearly independent set, and a row of 0’s implies a linearly dependent set).

Finding the Inverse:
• To calculate the inverse of a nonsingular \( nxn \) matrix, proceed as follows:
  o Form the matrix \([ A | I ]\)
  o Use elementary row operations to transform the above into \([ I | B ]\)
  o The resulting matrix \( B = A^{-1} \)
• Example: Find the inverse of \( A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \)
Solution:
\[
\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}
\]
Therefore, \( A^{-1} = \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix} \)

How can we check this?
• What can go wrong? Let’s try to find the inverse of \( A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \)
Solution:
\[
\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0.33 \\ 1 & -0.33 \end{bmatrix}
\]
We can’t form the identity matrix, therefore \( A \) does not have an inverse. It is not invertible, and with closer inspection we can see it is singular.

Properties of Matrix Inverses:
• For two \( nxn \) matrices \( A \) and \( B \), each of which has an inverse…
  o \( A^{-1} \) has an inverse, and \( (A^{-1})^{-1} = A \)
  o \( AB \) has an inverse, and \( (AB)^{-1} = B^{-1}A^{-1} \)
  o If \( k \) is a nonzero scalar, then \( kA \) has an inverse and \( (kA)^{-1} = (1/k)A^{-1} \)
  o \( A^T \) has an inverse, and \( (A^T)^{-1} = (A^{-1})^T \)
• Exercise: For \( A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \) and \( B = \begin{bmatrix} 2 & 3 \\ 6 & 7 \end{bmatrix} \), find \( AB, (AB)^{-1}, A^{-1}, B^{-1}, \) and \( B^{-1}A^{-1} \)

\textbf{Using Inverses to Solve Systems of Linear Equations:}

• If we have the system \( Ax = \bar{b} \), the a solution exists if \( A \) is invertible, and it would be \( \bar{x} = A^{-1}\bar{b} \)

• Exercise: If we know \( A = \begin{pmatrix} 7 & 4 \\ 5 & 3 \end{pmatrix} \) has an inverse \( A^{-1} = \begin{pmatrix} 3 & -4 \\ -5 & 7 \end{pmatrix} \), then what is the solution to the system given by \( 7x_1 + 4x_2 = 5, \ 5x_1 + 3x_2 = 2 \) ?

\textbf{Solution:}

\[ A\bar{x} = \bar{b} \Rightarrow \begin{pmatrix} 7 & 4 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ -5 & 7 \end{pmatrix} \begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ -11 \end{pmatrix} \]

\textbf{Hilbert Matrix:}

• The \( nxn \) Hilbert matrix is the matrix whose \( ij \)th entry is \( 1/(i + j - 1) \).

• The 3x3 Hilbert matrix is

\[ \begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{pmatrix} \]

• The above matrix has an inverse given by

\[ \begin{pmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{pmatrix} \]

• Note the size of the numbers in the matrix.

• If small changes in \( \bar{b} \) in the equation \( A\bar{x} = \bar{b} \) can lead to large changes in the solution \( \bar{x} \) then the matrix \( A \) is called \textbf{ill-conditioned}. 