Section 5.4 – Fundamental Theorem of Calculus

Taking Some Things For Granted:
• So far in Chapter 5 we have seen several concepts:
  Reimann Integration
  \[ \int_a^b f(x) \, dx = F(x)|_a^b = F(b) - F(a) \quad \text{and} \quad \int f(x) \, dx = F(x) + c. \]
• We have been taking for granted that our functions are ‘nice’ in that they are defined and continuous
• We would run into trouble, for example, if we tried to find the area for \( f(x) = \frac{1}{x} \) on \([0,5]\).

\[
\int_0^5 \frac{1}{x} \, dx = \ln x |_0^5 = \ln 5 - \ln 0
\]
• The fact that the above function is not continuous at \( x = 0 \) causes problems when trying to integrate it over that domain.

Fundamental Theorem, Part I:
• Assuming \( f(x) \) is continuous on a domain \([a,b]\), then we can integrate it, \( \int f(x) \, dx \), and this integral in turn is also continuous on \([a,b]\) and differentiable on \((a,b)\) and further \( \frac{d}{dx} \left[ \int f(x) \, dx \right] = f(x). \)
• This notation is slightly different than the book, which uses a definite integral of the form
  \[
  \frac{d}{dx} \left[ \int_a^x f(x) \, dx \right] = f(x).
  \]
• Example. Differentiate \( g(x) = \int_1^x \frac{1}{x+x^2} \, dx \)
Example. Differentiate \( h(x) = \int_0^2 \sqrt{1 + r^2} \, dr \)

**Fundamental Theorem, Part II:**

- We have already seen this informally, \( \int_a^b f(x) \, dx = F(x) \bigg|_a^b = F(b) - F(a) \) for \( F' = f \). Again, we just need to be sure that \( f \) is continuous on the domain \([a,b]\)

Example. Evaluate \( \int_0^4 (1 + 3y - y^2) \, dy \)

Example. Evaluate \( \int_0^{\pi/6} \csc x \cot x \, dx \)