Section 4.5 – Optimization Problems

- Example. Find two numbers whose difference is 100 and whose product is a minimum

What we know:
Two numbers \( x, y \).
Difference of \( x \) and \( y \) is 100, i.e. \( x - y = 100 \)
Product of \( x \) and \( y \) is a minimum, i.e. \( xy \) is a min

Solve:
Solving the first equation for \( x \) and substituting into the second we have
\[
y(100 + y) = 100y + y^2
\]
\[
(100y + y^2)' = 100 + 2y
\]
This has a critical point at \( y = -50 \)

Test:
Is this a maximum? \((100 + 2y)' = 2\), so the second derivative is always positive.
Hence, \( y = -50 \) corresponds to an absolute min.
With \( y = -50 \), \( x = 50 \).

Solution:
So two numbers whose difference is a 100 and whose product is a minimum are \(-50 \) and \(50 \).

- Example. Find the dimensions of a rectangle with area 1000 sq. meters whose perimeter is as small as possible.

What we know:
Area = \( lw = 1000 \)
Perimeter = \( 2l + 2w \)

Solve:
Solving the first equation for \( w \) and substituting into the second we find
\[
2 \frac{1000}{w} + 2w = 2000w^{-1} + 2w
\]
Differentiating with respect to \( w \) we find
\[
(2000w^{-1} + 2w)' = -2000w^{-2} + 2
\]
Setting the derivative equal to zero and solving for \( w \) we find
\[-2000w^{-2} + 2 = 0 \]
\[w^2 = 1000 \]
\[w \approx 31.6 \]

Test:
This corresponds to a minimum because \((-2000w^{-2} + 2)' = 4000w^{-3}\) evaluated at 31.6 is positive.

Solution:
So the rectangle would be 31.6 m by 31.6 m.
Example. A box with a square base and open top must have a volume of 32000 cubic cm. Find the dimensions of the box that minimize the amount of material used.

What we know:
- A box with a square base and open top has base \( s \) by \( s \), and height \( h \)
- Its volume is given by \( s^2h = 32000 \)
- The material used to construct it would be: Bottom = \( s^2 \), Top = none, Each side = \( sh \)
- The material used (total) would be \( s^2 + 4sh \)

Solve:
- Solving the first equation for \( h \) and substituting into the other, we find

\[
\frac{s^2 + 4s \left( \frac{32000}{s^2} \right)}{s^2} = s^2 + 128000s^{-1}
\]

- Taking the derivative and setting equal to zero, we find

\[
\left( s^2 + 128000s^{-1} \right)' = 2s - 128000s^{-2} = 0
\]

\[
s^3 = 64000 \quad \text{or} \quad s = 40
\]

Check:
- This corresponds to a minimum because \( (2s - 128000s^{-2})' = 2 + 256000s^{-3} \) is positive for \( s = 40 \).

Solution:
- So the dimensions of the box are: 40 cm square base and 20 cm height
Example. A rectangular storage container with an open top is to have a volume of 10 m$^3$. The length of its base is twice the width. Material for the base costs $10 per square meter. Material for the sides costs $6 per square meter. Find the cost of the materials for the cheapest such container.

What we know:
- Container has base $l$ by $w$, and height $h$
- The volume of the container is $lwh = 10$
- The length of the base is twice the width implies $l = 2w$

Solve:
Substituting the second equation into the first we find
\[(2w)wh = 10 \implies 2w^2h = 10\]
We want to minimize the cost of the box:
- Two of the sides areas measure $lh$, and the other two measure $wh$.
- Cost of the sides is $6(2h + 2wh)$
- The base area measures $lw$.
- Cost of the base is $10(lw)$

Using the equation $l = 2w$ to substitute, we have a cost of
\[12lh + 12wh + 10lw = 12(2w)h + 12wh + 10(2w)w = 36wh + 20w^2\]
Solving $2w^2h = 10$ for $h$ and plugging into the above eqn we find
\[36w \frac{5}{w^2} + 20w^2 = 180w^{-1} + 20w^2.\]
Differentiating this with respect to $w$ and setting equal to zero we find,
\[(180w^{-1} + 20w^2)' = -180w^{-2} + 40w = 0\]
\[4.5 = w^3\]
\[w \approx 1.65\]

Check:
This corresponds to a minimum when we look at the second derivative.

Solution:
So $h = \frac{5}{1.65^2} \approx 1.84$, and $l = \frac{10}{wh} = \frac{10}{1.65(1.84)} \approx 3.29$

So the box should measure $3.29l \times 1.65w \times 1.84h$ in meters
This box would cost about $36(1.65)(1.84) + 20(1.65)^2 = $163.75.

Q: Why do I say ‘about’?
A: Because we had to round the values for $h$, $w$ and $l$
Example. Find the point on the line $6x + y = 9$ that is closest to the point $(−3,1)$.

What we know:

The distance formula (squared) is given by

$$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$

For the point we are trying to get close to, the formula becomes

$$d^2 = (x + 3)^2 + (y - 1)^2$$

Solve:

Substituting $y = 9 - 6x$ into the above equation we find

$$d^2 = (x + 3)^2 + (8 - 6x)^2$$

Taking the derivative, setting it equal to zero and solving for $x$ we find

$$[(x + 3)^2 + (8 - 6x)^2]' = 2(x + 3) - 12(8 - 6x)$$

$$= 74x - 90$$

$$74x - 90 \equiv 0 \Rightarrow x = \frac{45}{37}$$

Check:

This is a minimum when we look at the second derivative.

Solution:

$$y(\frac{45}{37}) = 9 - 6\left(\frac{45}{37}\right) = \frac{63}{37}.$$ 

So the point that is closest is $\left(\frac{45}{37}, \frac{63}{37}\right)$. 