Section 3.6 – Hyperbolic Functions

Definitions:

- Recall the trig functions are related to the unit circle $x^2 + y^2 = 1$…

- In a similar way, hyperbolic functions are related to the hyperbolic function $x^2 - y^2 = 1$…

- They can be expressed in terms of linear combinations of exponential growth and decay.

- $\sinh x = \frac{e^x - e^{-x}}{2}$
  $\cosh x = \frac{e^x + e^{-x}}{2}$

- For the regular trig functions, sine and cosine give rise to tangent, cotangent, secant and cosecant.

- In a similar way, hyperbolic sine and hyperbolic cosine give rise to…

\[
\begin{align*}
\tanh x &= \frac{\sinh x}{\cosh x} \\
\csch x &= \frac{1}{\sinh x} \\
\sech x &= \frac{1}{\cosh x} \\
\coth x &= \frac{\cosh x}{\sinh x}
\end{align*}
\]
Graphs of Hyperbolic Sine and Cosine:

- \( \sinh x = \frac{e^x - e^{-x}}{2} \)

- \( \cosh x = \frac{e^x + e^{-x}}{2} \)
Properties of Hyperbolic Functions:

- \( \sinh(-x) = \frac{e^{-x} - e^{-(x)}}{2} = \frac{-e^x - e^{-x}}{2} = -\sinh x \)

- \( \cosh(-x) = \frac{e^{-x} + e^{-(x)}}{2} = \frac{e^x + e^{-x}}{2} = \cosh x \)

- Q: From our definitions of **even** and **odd**, how can we classify \( \sinh \) and \( \cosh \)?
  A: 

- Q: What can we then say about their symmetry?
  A: 

- \( \cosh^2 x - \sinh^2 x = 1 \)

- And the last two properties are:
  \( \sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y \)
  \( \cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y \)

Derivatives:

- \( \frac{d}{dx} \sinh x = \frac{d}{dx} \left( \frac{e^x - e^{-x}}{2} \right) = \frac{e^x + e^{-x}}{2} = \cosh x \).
  The derivative of \( \sinh \) is \( \cosh \).

- \( \frac{d}{dx} \cosh x = \frac{d}{dx} \left( \frac{e^x + e^{-x}}{2} \right) = \frac{e^x - e^{-x}}{2} = \sinh x \).
  The derivative of \( \cosh \) is \( \sinh \).

- \( \frac{d}{dx} \tanh x = \frac{d}{dx} \left( \frac{\sinh x}{\cosh x} \right) = \frac{\cosh x(\cosh x) - \sinh x(\sinh x)}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \text{sech}^2 x \)

- Q: Use your understanding of the chain rule and the property that \( \text{csch} x = 1/\sinh x \) to find \( \frac{d}{dx} \text{csch} x \).
  A:
Q: Use your understanding of the property \( \text{csch} \, x = \frac{2}{e^x - e^{-x}} \) to find \( \frac{d}{dx} \text{csch} \, x \).

A:

Q: Use your understanding of the chain rule and the property that \( \text{sech} \, x = \frac{1}{\cosh \, x} \) to find \( \frac{d}{dx} \text{sech} \, x \).

A:

Q: Use your understanding of the chain rule and the property that \( \coth \, x = \frac{\cosh \, x}{\sinh \, x} \) to find \( \frac{d}{dx} \coth \, x \).

A:

Inverse Hyperbolic Functions:

- There are functions that ‘undo’ the hyperbolic functions
  
  \[ y = \sinh^{-1} x \iff \sinh y = x \]
  
  \[ y = \cosh^{-1} x \iff \cosh y = x \quad y \geq 0 \]
  
  \[ y = \tanh^{-1} x \iff \tanh y = x \]

- Because hyperbolic functions are related to exponentials, their inverses are related to logs
  
  \[ \sinh^{-1} x = \ln(x + \sqrt{x^2 + 1}) \quad x \in R \]
  
  \[ \cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}) \quad x \geq 1 \]
  
  \[ \tanh^{-1} x = \frac{1}{2} \ln \left( \frac{1 + x}{1 - x} \right) \quad -1 < x < 1 \]
• Let’s try to see why the first equation works…
  We want to find the inverse of the function \( y = \sinh x \)
  As we usually proceed to find an inverse, we swap \( x \) and \( y \) and solve for \( y \)
  \[
  x = \sinh y = \frac{e^y - e^{-y}}{2}
  \]
  \[
  2x = e^y - e^{-y}
  \]
  \[
  2xe^y = e^{2y} - 1
  \]
  \[
  (e^y)^2 - 2x(e^y) - 1 = 0
  \]
  \[
  e^y = \frac{2x \pm \sqrt{4x^2 - 4(1)(-1)}}{2(1)} = x \pm \sqrt{x^2 + 1} \quad \text{by the quadratic formula.}
  \]
  So either \( e^y = x - \sqrt{x^2 + 1} \). But \( x - \sqrt{x^2 + 1} < 0 \) for all \( x \). This is not valid.
  or \( e^y = x + \sqrt{x^2 + 1} \Rightarrow y = \ln(x + \sqrt{x^2 + 1}) \)

• We can also differentiate any of the inverse trig functions, these formulas are in your book.