Section 3.3 – Derivative of Log and Exponential Functions

Derivative of the Exponential Function:

- Recall that \( e \) can be defined by the number so that \( \lim_{h \to 0} \frac{e^h - 1}{h} = 1 \).

- Using the definition of derivative, we find
  \[
  \frac{d}{dx}(e^x) = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \to 0} \frac{e^x(e^h - 1)}{h} = e^x \lim_{h \to 0} \frac{e^h - 1}{h} = e^x(1) = e^x.
  \]

- The derivative of the Natural Exponential Function is itself
  \[
  \frac{d}{dx}(e^x) = e^x
  \]

Derivative of General Exponential:

- To differentiate \( f(x) = a^x \) we need to know three things…
  1. \( f(x) = a^x = (e^{\ln a})^x = e^{(\ln a)x} \)
  2. \( \frac{d}{dx} e^x = e^x \)
  3. The chain rule

- We then find that
  \[
  \frac{d}{dx} a^x = \frac{d}{dx} e^{(\ln a)x} = e^{(\ln a)x} [\ln a] = (e^{\ln a})^x \ln a = a^x \cdot \ln a
  \]

Derivative of the Log Function:

- The relationship between the log and exponential functions is: .
- To differentiate \( f(x) = \log_a x \) we need to know three things…
  1. The relationship of: \( \log_a x = y \Leftrightarrow a^y = x \)
  2. \( \frac{d}{dx} (a^y) = a^y \ln a \)
  3. Implicit differentiation

- If we use implicit differentiation on the equation \( a^y = x \) we find
  \[
  \frac{d}{dx} (a^y) = \frac{d}{dx} x
  \]
  \[a^y \cdot \ln a \cdot y' = 1\]
  \[y' = \frac{1}{a^y \cdot \ln a} = \frac{1}{x \cdot \ln a}\]

- Therefore \( (\log_a x)' = \frac{1}{x \cdot \ln a} \).
Derivative of Natural Log Functions:

- From above, we have $(\log_a x)' = \frac{1}{x \cdot \ln a}$.

- For most practical purposes, we use the natural log, or a base of $a = e$. So we have $\frac{d}{dx} \ln x = \frac{1}{x}$

- So with the chain rule, we say: The derivative of the natural log of a function is one over that function times the derivative of that function.

Examples:

- Find the derivative of $y = \ln \left( \frac{x}{x+1} \right)$.
• Find the derivative of \( y = \frac{1 + \ln x}{1 + (\ln x)^2} \).

• Find the derivative and domain for \( f(x) = \frac{1}{1 + \ln x} \).