Section 7.1 – Sequences and Series

- Let's say you have a job that currently pays you $38,000 per year. Your employer gives you a cost of living raise of 2% every year you work there. Over the next several years, your salary will go up like this:

<table>
<thead>
<tr>
<th>Year</th>
<th>Increase</th>
<th>Salary</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td>38000</td>
</tr>
<tr>
<td>1</td>
<td>38000(0.02)</td>
<td>38760</td>
</tr>
<tr>
<td>2</td>
<td>38760(0.02)</td>
<td>39535.20</td>
</tr>
<tr>
<td>3</td>
<td>39535.20(0.02)</td>
<td>40325.90</td>
</tr>
</tbody>
</table>

etc.

Now how much will you be making at year 1 and a half?

The same you make year 1, which is 38760

- Because we are looking at the above situation with inputs as only 1, 2, 3, 4, etc. it would be more convenient to restrict our domain to positive integers only

- A sequence is a function where inputs (domain) are restricted to the positive integers only

- They look, act and feel just like functions do, but they do not have a continuous domain. Therefore their graphs look like a bunch of dots instead of a continuous line

- A sequence can be finite (are inputs only go up to a certain value) or infinite (we take all positive integers as our inputs, which go toward infinity)

- Because we are restricting the domain of our 'function', we think about the range in a different way as well. Now we call the function values 'terms' of the sequence

- Also instead of using the same notation for functions, i.e. \( f(x) \), we use sequence notation, which is usually \( a_n \), where you would think about it like you would functions, where you have \( a(n) \)

**Example.** Given \( a_n = (n-1)(n-2)(n-3) \) find the first 4 terms, \( a_{10} \) and \( a_{15} \)

First term = \( a_1 = (1-1)(1-2)(1-3) = 0 \)

Second term = \( a_2 = (2-1)(2-2)(2-3) = 0 \)

Third term = \( a_3 = (3-1)(3-2)(3-3) = 0 \)

Fourth term = \( a_4 = (4-1)(4-2)(4-3) = 6 \)

\( a_{10} = (10-1)(10-2)(10-3) = 504 \)

\( a_{15} = (15-1)(15-2)(15-3) = 2184 \)

**Example.** Find \( a_{80} \) of \( a_n = \left(1 + \frac{1}{n}\right)^2 \)

\( a_{80} = \left(1 + \frac{1}{80}\right)^2 = 1.025 \)
Predicting a Sequence:
- Predicting a definition from a given sequence can be fun, just realize it is possible to do it more than one way
- **Example.** Predict the nth term for the sequence 3, 9, 27, 81, 243…
  
  Remember, your inputs are: 1, 2, 3, 4…
  
  You want to get 3 from 1, 9 from 2, 27 from 3, 81 from 4, 243 from 5, etc
  
  \[ a_n = 3^n \]

- **Example.** Predict the nth term for the sequence \( \sqrt{2}, 2, \sqrt{6}, 2\sqrt{2}, \sqrt{10} \ldots \)
  
  It may be hard to tell at first glance, but convert the sequence to \( \sqrt{2}, \sqrt{4}, \sqrt{6}, \sqrt{8}, \sqrt{10} \ldots \)
  
  Then we can see \( a_n = \sqrt{2n} \)

- **Example.** Predict the nth term for the sequence \(-1, -4, -7, -10, -13\ldots\)
  
  You are first going to be overall negative for sure
  
  Then you are subtracting 3 (adding \(-3\)) every input
  
  A first choice might be \( a_n = -(n + 3) \). But our first term needs to be \(-1\)! This doesn't work at all!
  
  Think about this sequence as: \(-3 + 2, -6 + 2, -9 + 2, -12 + 2, -15 + 2\ldots\)
  
  \[ a_n = -3n + 2 \]

Recursive Sequences:
- For the previous example, we quickly saw that we were subtracting 3 every time. It would have been nice to write the sequence something like...
  
  \[ a_1 = -1 \]
  
  \[ a_2 = a_1 - 3 \]
  
  \[ a_3 = a_2 - 3 \]
  
  \[ \vdots \]
  
  \[ a_n = a_{n-1} - 3 \]

  This is what is known as a recursive sequence, where you define each term based on the one before it. The advantage is that it is easy to see and build a recursive sequence, the disadvantage is that you cannot jump to a particular term

- **Example.** \( a_1 = 4, a_{n+1} = 1 + \frac{1}{a_n} \)
  
  \[ a_1 = 4 \]
  
  \[ a_2 = 1 + \frac{1}{4} = \frac{5}{4} \]
  
  \[ a_3 = 1 + \frac{1}{5/4} = \frac{9}{5} \]
  
  \[ a_4 = 1 + \frac{1}{9/5} = \frac{14}{9} \]

  But what would be the formula to find \( a_{20} \)?
  
  \[ a_{20} = 1 + \frac{1}{a_{19}} \]. You need to know \( a_{19} \), to get \( a_{19} \) you need \( a_{18} \), etc
Series:
- Sometimes it is convenient to add all the terms in a sequence. We denote this with 'S' and call it a series.
- An infinite sequence gives rise to an infinite series, and a finite series (also called an nth partial sum, $S_n$) refers to the fact that it is capped at some point.
  
  \[ S_n = a_1 + a_2 + a_3 + \ldots + a_n \]
- A series (or partial sum) has another notation using a $\Sigma$. Where have we seen this symbol before? With this symbol you can use a starting point (underneath) and an ending point (on top).

For example, $\sum_{i=5}^{10} a_i = a_5 + a_6 + a_7 + a_8 + a_9 + a_{10}$
- It is standard practice to use letters like $i, j$ and $k$ for your indices or inputs
- Example. Find $\sum_{i=1}^{6} \frac{1}{2i+1}$

\[
\sum_{i=1}^{6} \frac{1}{2i+1} = \frac{1}{2(1)+1} + \frac{1}{2(2)+1} + \frac{1}{2(3)+1} + \frac{1}{2(4)+1} + \frac{1}{2(5)+1} + \frac{1}{2(6)+1} = 0.955
\]

Some Very Cool Sequences and Series:
- First, if you haven't seen it already, the factorial is defined as

  \[
  0! = 1 \\
  1! = 1 \\
  2! = 2 \cdot 1 \\
  3! = 3 \cdot 2 \cdot 1 \\
  4! = 4 \cdot 3 \cdot 2 \cdot 1 \\
  10! = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1
  \]

  etc.

- Let's look at the series given by $\sum_{k=0}^{n} \frac{1}{k!}$
  
  For $n = 5$, we get $\sum_{k=0}^{5} \frac{1}{k!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} = 1 + 1 + 0.5 + 0.16 + 0.0416 + 0.0083 = 2.7166$
  
  Does this value look familiar? What is this value?
  
  In fact, the higher $n$ we take the closer we get to this particular value

- Let's look at the series given by $\sum_{k=0}^{n} \frac{(-1)^k}{2k+1}$
  
  For $n = 15$ we get

  \[
  \sum_{k=0}^{15} \frac{(-1)^k}{2k+1} = \frac{(-1)^0}{2(0)+1} + \frac{(-1)^1}{2(1)+1} + \frac{(-1)^2}{2(2)+1} + \frac{(-1)^3}{2(3)+1} + \ldots + \frac{(-1)^{15}}{2(15)+1}
  \]

  \[
  = 1 - \frac{1}{3} - \frac{1}{5} - \frac{1}{7} - \frac{1}{9} - \frac{1}{11} - \frac{1}{13} - \frac{1}{15} \approx 0.769788348
  \]

  Let's multiply this number by 4. We get 3.079153394
  
  Does this look familiar?

- Actually, most famous irrational numbers can be written in terms of a series of some kind