Section 4.2 – Exponential Functions and Graphs

What Would You Rather Have...
• $1 million, or double your money every day for 31 days starting with 1 cent?

<table>
<thead>
<tr>
<th>Day</th>
<th>Cents</th>
<th>Day</th>
<th>Cents</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>16</td>
<td>65536</td>
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<td>17</td>
<td>131072</td>
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<td>4</td>
<td>18</td>
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<td>8</td>
<td>19</td>
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<td>16</td>
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<td>24</td>
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<td>9</td>
<td>512</td>
<td>25</td>
<td>33554432</td>
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<td>26</td>
<td>67108864</td>
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<tr>
<td>11</td>
<td>2048</td>
<td>27</td>
<td>134217728</td>
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<tr>
<td>12</td>
<td>4096</td>
<td>28</td>
<td>268435456</td>
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<tr>
<td>13</td>
<td>8192</td>
<td>29</td>
<td>536870912</td>
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<tr>
<td>14</td>
<td>16384</td>
<td>30</td>
<td>1073741824</td>
</tr>
<tr>
<td>15</td>
<td>32768</td>
<td>31</td>
<td>2147483648</td>
</tr>
</tbody>
</table>

• On day 31, you will have 2,147,483,648 cents, or $21,474,836.48
• How can we express this with an equation?
  \[ \# \text{cents}(\text{day}) = 2^{\text{day}} \]
  \[ 2^{31} = 2,147,483,648 \]
• When do you surpass $1 million, or \( 10^8 \) cents?
  From the chart of values above, you can see this happens between day 26 and day 27

Some Background:
• Evaluate
  \[ 3^7 \approx 31.54 \]
  \[ 3^{-\sqrt{15}} \approx 0.01419 \]
  \[ 3^{-\frac{1}{3}} \approx 1.49804 \]
• Example. Find \( \left( \frac{1}{e^3} \right)^2 \)
  \[ \left( \frac{1}{e^3} \right)^2 = \frac{1}{e^6} \approx 2.48 \times 10^{-3} = 0.00248 \]

Form of the Exponential Function:
• \( y = kb^t \)
• Variables are \( y \) and \( t \), \( y \) depends on \( t \)
• \( b \) is the base ( \( b > 0, b \neq 1 \) )
• \( k \) is the initial quantity (when \( t = 0 \))
• A commonly used base is \( e = 2.7182818284… \)
Rules for Exponentials:

\[ b^x b^y = b^{x+y} \]
\[ (b^x)^y = b^{xy} \]
\[ b^{-x} = \frac{1}{b^x} \]

Graphs of Exponentials:

- If the base is greater than 1, you have exponential growth
- If the base is less than 1, you have exponential decay
- If \( k > 0 \), the function will always be positive. It will tend to 0 (for decay) and infinity (for growth)
- If \( k < 0 \), the function will always be negative (it is flipped upside down)
- The \( y \) intercept is \( k \)

- If you have an equation that is exponential in form, but are adding or subtracting a constant
  \[ y = k b^t + c \], this is a shift up (\( c > 0 \)) or down (\( c < 0 \)) by \( c \)
  \[ y = k b^{(t+c)} \], this is a shift left (\( c < 0 \)) or right (\( c > 0 \)) by \( c \)

**Example. Graph** \( y = 3^{-x} \)

\[ y = \left(\frac{1}{3}\right)^x \]
\( b = 1/3 \)  
Exponential decay  
\( k = 1 \)  
Crosses at (0,1)

**Example. Graph** \( f(x) = 2 - e^{-x} \)

\[ f(x) = 2 + (-1)\left(\frac{1}{e}\right)^x \]
This has a shift up of 2  
\( k = -1 \) (flipped upside down)  
and \( b = 1/e \approx 0.37 \)
**Exponential Equations Arise When:**
- You add the same percentage to a quantity each fixed time period
- You multiply a quantity by the same amount each fixed time period
- For the money example, we were adding 100% each day, or multiplying by 2

**Another Example, Simple Interest:**
- Suppose your savings account earns 1.25% interest per month, and you start with $250. Write an equation that represents this problem.

How can we make an equation out of this?

\[ y(t) = 250(1.0125)^t \]

- Does this represent growth or decay? Growth
- How much do you earn in the first month? \( y(1) = 250(1.0125) - 250 \approx 3.13 \)

**The Relationship between the Rate of Growth and \( b \):**
- Sometimes exponential relationships are given as rate of growth problems
- Similar to above, the rate of growth of the equation is given as 1.25%
- So \( b \) is the base value in the equation \( y(t) = k \ b^t \), and \( r \) is the rate of growth (or decay)
- Note that \( r \) is usually given as a percentage, and you must convert it to decimal before using
- \( r = 1 + r, r = b - 1 \)
- \( r \) can be negative or positive, but \( b \) must be positive and not equal to 1
- In the double your money problem
  \( b = 2, \) and \( r = 100\%. \) \( r = 2 - 1 = 1 = 100\%. \) \( b = 1 + r = 2 \)

**Example 1:**
- Determine if the following set of data is exponential by taking divisions of successive \( y \) values

<table>
<thead>
<tr>
<th>( t )</th>
<th>( y )</th>
<th>( y2/y1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>18</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>54</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>162</td>
<td>3</td>
</tr>
</tbody>
</table>

- So this data seems to be exponential
- What equation best represents this data?
  \( b = y2/y1 = 3 \)
  \( k = y(0) = 2 \)
  \( y = 2 \cdot 3^t \)
Example 2:

• Determine if the following set of data is exponential

<table>
<thead>
<tr>
<th>t</th>
<th>y</th>
<th>y2 / y1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>75</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>73.5</td>
<td>0.98</td>
</tr>
<tr>
<td>2</td>
<td>72.03</td>
<td>0.98</td>
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<tr>
<td>3</td>
<td>70.5894</td>
<td>0.98</td>
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<tr>
<td>4</td>
<td>69.17761</td>
<td>0.98</td>
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<tr>
<td>5</td>
<td>67.79406</td>
<td>0.98</td>
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<tr>
<td>6</td>
<td>66.43818</td>
<td>0.98</td>
</tr>
</tbody>
</table>

• So the data seems to be exponential with \( b = 0.98 \) and \( k = 75 \)

**Compound Interest:**

• \( A = P \left(1 + \frac{r}{n}\right)^{nt} \)

  \( A \) is the amount of money at any time \( t \)
  \( P \) is the principal (initial investment)
  \( t \) number of years
  \( r \) is the interest rate
  \( n \) is the number of times compounded per year

• Example. Find the balance for $12,500 invested at 3\% for 3 years compounded quarterly

  \[ A = 12500 \left(1 + \frac{0.03}{4}\right)^{4(3)} = 13,672.59 \]